BENG 221 Mathematical Methods in Bioengineering

Fall 2017

Midterm

NAME: SOLUTIONS

- Open book, open notes.
- 80 minutes limit (end of class).
- No communication other than with instructor and TAs.
- No computers or internet, except for access to posted class materials.
### Table 1: Laplace and Fourier Transforms

<table>
<thead>
<tr>
<th>( u(t) )</th>
<th>( U(s) )</th>
<th>( U(j\omega) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta(t) )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{s} )</td>
<td>( \frac{1}{j\omega} )</td>
</tr>
<tr>
<td>( e^{-at} )</td>
<td>( \frac{1}{s+a} )</td>
<td>( \frac{1}{j\omega+a} )</td>
</tr>
<tr>
<td>( u(t-t_0) )</td>
<td>( e^{-st_0} U(s) )</td>
<td>( e^{-j\omega t_0} U(j\omega) )</td>
</tr>
<tr>
<td>( \frac{du}{dt} )</td>
<td>( sU(s) - u(0) )</td>
<td>( j\omega U(j\omega) )</td>
</tr>
<tr>
<td>( \int_{-\infty}^{t} u(t_0) , dt_0 = u(0) + \int_{0}^{t} u(t_0) , dt_0 )</td>
<td>( \frac{1}{s} U(s) )</td>
<td>( \frac{1}{j\omega} U(j\omega) )</td>
</tr>
<tr>
<td>( \int_{-\infty}^{t} h(t-t_0) f(t_0) , dt_0 = u(0) + \int_{0}^{t} h(t-t_0) f(t_0) , dt_0 )</td>
<td>( H(s) \cdot F(s) )</td>
<td>( H(j\omega) \cdot F(j\omega) )</td>
</tr>
</tbody>
</table>

### Table 2: Green's Functions for Diffusion in 1-D

<table>
<thead>
<tr>
<th>B.C.</th>
<th>( G(x, t; x_0, t_0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = 0 )</td>
<td>( x = L )</td>
</tr>
<tr>
<td>( t &gt; t_0 )</td>
<td>( t &gt; t_0 )</td>
</tr>
</tbody>
</table>

\[
\mathcal{N}(x_0, \sqrt{2D(t-t_0)}) = \frac{1}{\sqrt{4\pi D(t-t_0)}} \exp\left(-\frac{(x-x_0)^2}{4D(t-t_0)}\right)
\]

\[
\mathcal{N}(x_0, \sqrt{2D(t-t_0)}) - \mathcal{N}(-x_0, \sqrt{2D(t-t_0)})
\]

\[
\mathcal{N}(x_0, \sqrt{2D(t-t_0)}) + \mathcal{N}(-x_0, \sqrt{2D(t-t_0)})
\]

\[
\sum_{k=1}^{\infty} \frac{2}{L} \sin\left(\frac{k\pi}{L} x_0\right) \sin\left(\frac{k\pi}{L} x\right) \exp\left(-\frac{(k\pi)^2}{L^2} D(t-t_0)\right)
\]

\[
\sum_{k=0}^{\infty} \frac{2}{L} \sin\left(\frac{(k + \frac{1}{2})\pi}{L} x_0\right) \sin\left(\frac{(k + \frac{1}{2})\pi}{L} x\right) \exp\left(-\frac{(k + \frac{1}{2})^2\pi^2}{L^2} D(t-t_0)\right)
\]

\[
\sum_{k=0}^{\infty} \frac{2}{L} \cos\left(\frac{(k + \frac{1}{2})\pi}{L} x_0\right) \cos\left(\frac{(k + \frac{1}{2})\pi}{L} x\right) \exp\left(-\frac{(k + \frac{1}{2})^2\pi^2}{L^2} D(t-t_0)\right)
\]

\[
\frac{1}{L} + \sum_{k=1}^{\infty} \frac{2}{L} \cos\left(\frac{k\pi}{L} x_0\right) \cos\left(\frac{k\pi}{L} x\right) \exp\left(-\frac{k^2\pi^2}{L^2} D(t-t_0)\right)
\]
Problem 1 (20 points): Short answer problems. Provide brief explanations (no lengthy derivations!) for each problem.

1. (5 points): Among all the eigenvalue-eigenvector pairs obtained by singular value decomposition of a matrix containing multi-dimensional data samples, which one explains most of the variance in the data?

   The one with the largest eigenvalue (principal component).

2. (5 points): Give an example that demonstrates why diffusion over a bounded interval is not space-invariant.

   A source acting on a zero-value boundary produces no response, whereas a source acting away from the boundary does not.

3. (5 points): To arrive at the diffusion transfer function $H(k, s)$, does it matter in which order the Fourier and Laplace transforms are applied to the diffusion equation, and why?

   No: Fourier and Laplace are linear transforms, so the order in which they are applied doesn't matter.

4. (5 points): Solve the following integral:

   $$
   \int_{-\infty}^{+\infty} \delta(x - x_0) \exp(-jkx) \, dx = \exp(-jkx_0)
   $$

   Delta-Dirac picks the integrand at its center, $x_0$. 
Problem 2  (30 points): Consider a two-segment lumped model of diffusion along a passive cable of length $L$, with line resistivity $r$ and line capacitance $c$, and with zero-voltage boundary conditions on both ends, as shown below. The length of each of the two segments is $\Delta x = L/2$.

$$
\begin{align*}
v_0 &= 0 \\
 v_1 & \\
 r\Delta x & \quad r\Delta x \\
 v_2 &= 0 \\
 c\Delta x & \quad c\Delta x
\end{align*}
$$

1. (10 points): Write the ordinary differential equation governing the dynamics of the voltage $v_1(t)$ at the center of the cable. Is this ODE homogeneous or inhomogeneous?

KCL @ $v_1$ node:

$$
c\Delta x \frac{dv_1}{dt} = -\frac{1}{r\Delta x} \cdot v_1 - \frac{1}{r\Delta x} \cdot v_1
$$

$$
\Rightarrow \quad \frac{dv_1}{dt} = -\frac{2}{rc\Delta x^2} \cdot v_1 = -\frac{8}{rcL^2} \cdot v_1
$$

$$
\Rightarrow \quad \frac{dv_1}{dt} = -\frac{8D}{L^2} \cdot v_1 \quad \text{with} \quad D = \frac{1}{rc} \quad \text{diffusivity}
$$

Homogeneous \quad (\text{zero source})
2. (10 points): Show that the solution to this ODE is given by a decaying exponential over time, \( v_1(t) = A \exp(-t/\tau) \). Identify the amplitude constant \( A \) and the time constant \( \tau \) in terms of the initial condition \( v_1(0) \), cable length \( L \), and diffusivity \( D \).

\[
v_1(t) = v_1(0) \cdot e^{-\frac{8D}{L^2} t} = A \cdot e^{-\frac{1}{\tau} t}
\]

with

\[
\begin{align*}
A &= v_1(0) \\
\tau &= \frac{L^2}{8D}
\end{align*}
\]

3. (10 points): Compare the time constant \( \tau \) that you obtained in part 2 for this two-segment lumped approximation of the passive cable, with the time constant \( \tau_\infty \) for the infinite-segment continuum limit, corresponding to the dominant (first) term in the eigenmode series of the homogeneous solution for the cable with same diffusivity \( D \), same total length \( L \), and same zero-voltage boundary conditions on both ends.

First term: \( \psi(x,t) = A_1 \sin\left(\frac{\pi x}{L}\right) e^{-\left(\frac{\pi}{L}\right)^2 D t} \)

\[
\Rightarrow \tau_\infty = \frac{L^2}{\pi^2 D}
\]

Slightly (~15%) faster than \( \tau \)
Problem 3  (50 points): Here we will consider diffusion of estrogen through tissue into the bloodstream. The diffusivity $D$ is uniform across the tissue spanning length $L$ from the skin at $x = 0$, to the vasculature at $x = L$. The skin is completely impermeable to estrogen at $x = 0$. The vasculature completely absorbs any estrogen at $x = L$. A subcutaneous patch implanted at $x = x_0$ continuously supplies a constant but infinitely concentrated source of estrogen into the tissue: $f(x, t) = q_0 \delta(x - x_0)$ where $q_0$ is a constant, and $\delta(\cdot)$ is the delta-Dirac function. Initially the estrogen concentration $u(x, t)$ is given by:

$$
\begin{align*}
    u(x, 0) = g(x) &= \begin{cases} 
    \frac{q_0}{D} (L - x_0) & \text{for } 0 \leq x \leq x_0 \\
    \frac{q_0}{D} (L - x) & \text{for } x_0 < x \leq L
    \end{cases}
\end{align*}
$$

1. (5 points): Write the partial differential equation governing the estrogen concentration $u(x, t)$ in the tissue. Express initial and boundary conditions.

$$
\begin{align*}
    \frac{\partial u}{\partial t} &= D \frac{\partial^2 u}{\partial x^2} + f(x, t) \\
    \begin{cases}
        u(x, 0) = g(x) & \text{I.C.} \\
        \frac{\partial u}{\partial x} (0, t) = 0 & \text{flux b.c. @ 0} \\
        u(L, t) = 0 & \text{value b.c. @ L}
    \end{cases}
\end{align*}
$$

with the given source $f(x, t)$ and I.C. $g(x)$

2. (15 points): Solve for a particular solution $u_p(x)$ for the estrogen concentration in the tissue at steady state.

$$
\begin{align*}
    0 &= D \frac{d^2 u_p}{dx^2} + q_0 \delta(x-x_0) & \text{independent of time} \\
    \Rightarrow \frac{d^2 u_p}{dx^2} &= -\frac{q_0}{D} \delta(x-x_0) \\
    \Rightarrow \frac{du_p}{dx} &= -\frac{q_0}{D} \int \delta(x-x_0) \, dx = \begin{cases} 
        c & \text{for } 0 \leq x < x_0 \\
        c - \frac{q_0}{D} (x-x_0) & \text{for } x_0 < x \leq L
    \end{cases}
\end{align*}
$$

flux b.c. @ $x=0$ : $c = 0$

$$
\Rightarrow u_p(x) = \int \frac{du_p}{dx} \, dx = \begin{cases} 
        c^2 & \text{for } 0 \leq x \leq x_0 \\
        c^2 - \frac{q_0}{D} (x-x_0) & \text{for } x_0 \leq x \leq L
    \end{cases}
$$
\[ \text{VALUE B.C. @ L: } c' - \frac{q^o}{D} (L-x_0) = 0 \]

\[ \Rightarrow M_p(x) = \begin{cases} 
\frac{q^o}{D} (L-x_0) & \text{for } 0 \leq x < x_0 \\
\frac{q^o}{D} (x-x_0) = \frac{q^o}{D} (L-x) & \text{for } x_0 \leq x \leq L
\end{cases} 
\]

\[ = g(x) ! \]

3. (5 points): Write the homogeneous problem, with homogeneous partial differential equation and boundary conditions.

\[ \frac{\partial^2 \mu_H}{\partial t^2} - D \frac{\partial^2 \mu_H}{\partial x^2} ; \quad \begin{cases} 
\frac{\partial \mu_H}{\partial x}(0,t) = 0 & \text{FLUX B.C. @ 0} \\
\mu_H(L,t) = 0 & \text{VALUE B.C. @ L}
\end{cases} \]

4. (10 points): Find the eigenmode decomposition for the general solution of the homogeneous problem.

\[ \mu_H(x,t) = \sum_{k=0}^{\infty} A_k \phi_k(x) e^{-D\lambda_k t} \]

\[ \begin{cases} \text{FLUX B.C. @ 0} \\
\text{VALUE B.C. @ L}
\end{cases} ; \quad \lambda_k = \left( \frac{(k+\frac{1}{2})\pi}{L} \right)^2 
\]

\[ \phi_k(x) = \cos \left( \frac{(k+\frac{1}{2})\pi x}{L} \right) \]
5. (15 points): Solve for the estrogen concentration in the tissue over time \( u(x, t) \) from the given initial conditions.  

*Hint*: KISS (Keep It Simple & Stupid). Don’t get alarmed if your answer appears too simple. But do get alarmed if you’re wielding through pages of integrals...

\[
\mu(x, t) = \mu_p(x) + \mu_H(x, t) \\
= \mu_p(x) + \sum_{k=0}^{\infty} A_k \Phi_k(x) e^{-D_k t}
\]

I.C.: \( g(x) = \mu(x, 0) = \mu_p(x) + \sum_{k=0}^{\infty} A_k \Phi_k(x) \)

\( A_k : \sum_{k=0}^{\infty} A_k \Phi_k(x) = g(x) - \mu_p(x) \)

\[= 0 \]

\( \Rightarrow A_k = 0 \) for all \( k \)

\( \Rightarrow \mu(x, t) = \mu_p(x) \)