BENG 221 Mathematical Methods in Bioengineering

Fall 2014

Midterm

NAME: SOLUTIONS

- Open book, open notes.
- 80 minutes limit (end of class).
- No communication other than with instructor and TAs.
- No computers or internet, except for access to posted class materials.
Problem 1  (30 points): Short answer problems. Provide brief explanations (no lengthy derivations!) for each problem.

1. (5 points): A random walk process with random step size of standard deviation \( \Delta x \) and step time interval \( \Delta t \) is characterized with a diffusivity \( D \). How does the diffusivity \( D \) change as the random step size increases by a factor two (i.e., \( \Delta x \) becomes \( 2\Delta x \))?

\[ \text{D increases } 4 \times \quad (D \text{ becomes } 4D) \quad \text{because diffusivity scales as } \frac{\Delta x^2}{\Delta t} \]

2. (5 points): Conversely, how does the diffusivity \( D \) change as the step time interval increases by a factor two (i.e., \( \Delta t \) becomes \( 2\Delta t \))?

\[ \text{D halves } \quad (D \text{ becomes } \frac{D}{2}) \quad \text{similarly} \]

3. (10 points): Find the radial frequency \( \omega \) of wave oscillations of a free electron with energy \( E \).

\[ \omega = \frac{E}{\hbar} = \frac{2\pi E}{\hbar} \quad \text{from the time-dependent part } e^{-\frac{ieE}{\hbar}t} \]

in separation of variables of the Schrödinger equation \( \frac{\hbar}{i} \frac{\partial \Psi}{\partial t} = (-\frac{\hbar^2}{2m} \nabla^2 + V) \Psi = E \Psi \).

Also consistent with \( E = \hbar \omega = \hbar \frac{2\pi}{T} \) (Planck).

4. (10 points): Find the maximum time step \( \Delta t \) beyond which Euler numerical integration of the ODE \( \frac{dx}{dt} = -x/t \) becomes unstable (i.e., gives unbounded results).

\[ \Delta t_{\text{max}} = 2 \tau \quad \text{because } \ x(t + \Delta t) = x(t) + \Delta t \left( -\frac{x(t)}{\tau} \right) = (1 - \frac{\Delta t}{\tau}) x(t) \]

then \( \left| \frac{x(t + \Delta t)}{x(t)} \right| = \left| 1 - \frac{\Delta t}{\tau} \right| > 1 \quad \text{for } \Delta t > 2 \tau \)

leading to unbounded \( x(t) \) as \( t \to \infty \).
Problem 2  (25 points): Consider an electrically excitable cell as shown below. The cell has membrane capacitance $C_m$ and leak conductance $g_i$. The extracellular space has leak conductance $g_e$. Both intracellular and extracellular potentials are initially zero. At time zero, a constant current electrode current $I_{\text{elec}}(t) = I_0$ is injected into the extracellular space.

1. (5 points): Write the differential equations and initial conditions governing the dynamics of the intracellular and extracellular potentials, $v_i(t)$ and $v_e(t)$.

$$g_i v_i = C_m \frac{d}{dt} (v_e - v_i) = -g_e v_e + I_{\text{elec}}(t)$$

where $I_{\text{elec}}(t) = I_0$ \hspace{1cm} ($t \geq 0$)

I. C. : $v_i(0) = v_e(0) = 0$

2. (15 points): Find the intracellular potential $v_i(t)$ over time. You may use Laplace or any method of your choice.

\begin{align*}
\text{Laplace:} & \quad v_i(t) \rightarrow \tilde{v}_i(s) \quad \frac{1}{s} \tilde{v}_i \rightarrow s \tilde{v}_i \quad (v_i(0) = 0) \\
& \quad v_e(t) \rightarrow \tilde{v}_e(s) \quad \frac{1}{s} \tilde{v}_e \rightarrow s \tilde{v}_e \quad (v_e(0) = 0) \\
I_{\text{elec}}(t) = I_0 \rightarrow I_0 \cdot \frac{1}{s} \quad (t \geq 0)
\end{align*}
\[
g_i \ddot{v_i} = C_m s (\ddot{v_e} - \ddot{v_i}) = -g e \ddot{v_e} + \frac{1}{s} I_0
\]

Eliminating \( \ddot{v_e} \):
\[
g e \ddot{v_e} = -g_i \ddot{v_i} + \frac{1}{s} I_0
\]

\[
\Rightarrow g_i g e \ddot{v_i} = C_m I_0 - (g_i + g e) C_m s \ddot{v_i}
\]

\[
\ddot{v_i} = \frac{C_m I_0}{(g_i + g e) C_m s + g i g e} = \frac{I_0}{g i + g e} \cdot \frac{1}{s + \frac{g i g e}{g i + g e} C_m}
\]

Inverse Laplace:
\[
\frac{1}{s + a} \Rightarrow e^{-at}, \quad t \geq 0
\]

\[
\Rightarrow v_i(t) = \frac{I_0}{g i + g e} \cdot e^{-\frac{g i g e}{g i + g e} C_m t}, \quad t \geq 0
\]

3. (5 points): Find the minimum level of electrode current \( I_0 \) to be injected in order for the intracellular potential \( v_i(t) \) to reach the threshold \( V_{th} \) at which the cell generates an action potential. At what time does the action potential happen? Does the duration of the electrode current matter, and how?

\( v_i(t) \) is maximum at \( t=0 \), regardless of duration of electrode current

\[\Rightarrow \text{Action potential happens at} \quad t = \frac{I_0}{g_i + g e} \geq V_{th}, \quad \text{or} \quad I_0 \geq (g_i + g e) V_{th} \]

\((I_{\text{min}} = (g_i + g e) V_{th})

\((*) \) AFTER the step in current \( I_0 \) at time \( t=0 \).
Problem 3  (45 points): An athlete initially at rest starts to exercise. The body is covered with thermally insulating material. Underneath the skin \((x = 0)\) is muscle tissue of thickness \(L\), interfacing on the other end \((x = L)\) with vasculature. The thermal conductivity of the muscle tissue is \(k_0\), and the vasculature conducts heat to maintain the tissue interface at a constant temperature \(T_0\). Specific heat of the muscle tissue is \(c\), and mass density is \(\rho\). Once starting to exercise \((t \geq 0)\), the athlete burns calories (Joules) uniformly in the muscle tissue at constant rate, with heat generation \(Q(x, t) = Q_0\).

1. (5 points): Write the partial differential equation governing temperature \(u(x, t)\) in the muscle tissue. Express initial and boundary conditions.

\[
\frac{\rho c}{\partial t} = k_0 \frac{\partial^2 u}{\partial x^2} + Q_0
\]

I.C.: \(u(x, 0) = T_0\)

B.C.: \(\begin{cases} \frac{\partial u}{\partial x}(0, t) = 0 \\ u(L, t) = T_0 \end{cases}\)

2. (10 points): Solve for a particular solution \(u_p(x)\) for the temperature in the tissue at steady state.

Steady state: \(\frac{\rho c}{\partial t} \to 0\) for \(t \to \infty\)

\(\Rightarrow k_0 \frac{d^2 u_p}{dx^2} + Q_0 = 0\) with B.C. \(\begin{cases} \frac{du_p}{dx}(0) = 0 \\ u_p(L) = T_0 \end{cases}\)

\[
\frac{du_p}{dx} = -\frac{Q_0}{k_0} \int dx = -\frac{Q_0}{k_0} x + a
\]

B.C.: \(\frac{du_p}{dx}(0) = a = 0\)

\[u_p = -\frac{Q_0}{k_0} \int x \, dx = -\frac{Q_0}{2k_0} x^2 + b\]

B.C.: \(u_p(L) = -\frac{Q_0 L^2}{2k_0} + c = T_0\)

\(\Rightarrow u_p(x) = T_0 - \frac{Q_0}{2k_0} (x^2 - L^2)\)
3. (5 points): Write the homogeneous problem, with homogeneous partial differential equation and boundary conditions. You may combine constants into a thermal diffusivity \(D\).

\[
\frac{\partial u_H}{\partial t} = D \frac{\partial^2 u_H}{\partial x^2} \quad \text{with} \quad D = \frac{K_0}{c_0}
\]

B.C.: \[
\begin{align*}
\frac{\partial u_H}{\partial x} (0, t) &= 0 \quad \text{(flux)} \\
\phi_H (L, t) &= 0 \quad \text{(value)}
\end{align*}
\]

4. (10 points): Find the eigenmode decomposition for the general solution of the homogeneous problem.

Separation of variables: \(u_H(x, t) = \phi(x) \cdot G(t)\)

- Time: \[
\frac{dG}{dt} = -DG \quad \Rightarrow \quad G(t) = G(0) \cdot e^{-Dt}
\]

- Space: \[
\frac{d^2 \phi}{dx^2} + \lambda \phi = 0 \quad \Rightarrow \quad \phi(x) = A \cos(\sqrt{\lambda}x) + B \sin(\sqrt{\lambda}x)
\]

B.C. @ \(x=0\): \[
\frac{d\phi}{dx} (0) = 0 \quad \Rightarrow \quad B = 0
\]

B.C. @ \(x=L\): \[
\phi(L) = 0 \quad \Rightarrow \quad \cos(\sqrt{\lambda}L) = 0 \quad \Rightarrow \quad \sqrt{\lambda} = (n+\frac{1}{2}) \frac{\pi}{L}
\]

\(n = 0, 1, 2, \ldots\)

Summation:

\[
\Rightarrow \quad u_H(x, t) = \sum_{m=0}^{\infty} A_n \cos\left(\left(n+\frac{1}{2}\right) \frac{\pi x}{L}\right) e^{-D\left(n+\frac{1}{2}\right)^2 \frac{\pi^2 t}{L^2}}
\]
5. (15 points): Solve for the temperature in the tissue over time $u(x, t)$ from initial conditions.

$$u(x, t) = u_p(x) + u_H(x, t)$$

I. C. : $u(x, 0) = T_0 = u_p(x) + u_H(x, 0)$

$$= u_p(x) + \sum_{n=0}^{\infty} A_n \cos\left(\frac{n+\frac{1}{2}}{L} \frac{\pi x}{L}\right)$$

$$\Rightarrow A_n = \frac{2}{L} \int_0^L \left( T_0 - u_p(x) \right) \cos\left(\frac{n+\frac{1}{2}}{L} \frac{\pi x}{L}\right) dx$$

$$= \frac{Q_0}{K_0} \int_0^L \left( \frac{x^2}{L} - L \right) \cos\left(\frac{n+\frac{1}{2}}{L} \frac{\pi x}{L}\right) dx$$

$$= -2 L^2 \frac{Q_0}{K_0} \frac{(-1)^n}{(n+\frac{1}{2}) \pi^3} \quad \text{from lecture notes (or see next page)}$$

$$\Rightarrow u(x, t) = T_0 - \frac{1}{2} \frac{Q_0}{K_0} (x^2 - L^2) - 2L^2 \frac{Q_0}{K_0} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+\frac{1}{2}) \pi^3} e^{-D\left(\frac{(n+\frac{1}{2}) \pi}{L}\right)^2 t}$$
\[ A_m = \frac{Q_o}{K_0} \left( \frac{x^2}{L} - L \right) \cos \left( \left( \frac{1}{2} + m \right) \frac{\pi x}{L} \right) \int_0^L dx \]

Partial integration twice:

\[ A_m = \frac{Q_o}{K_0} \left( \frac{x^2}{L} - L \right) \left( \sin \left( \left( \frac{1}{2} + m \right) \frac{\pi x}{L} \right) \right)_0^L - \int_0^L \frac{2x}{L} \sin \left( \left( \frac{1}{2} + m \right) \frac{\pi x}{L} \right) dx \]

\[ = \frac{Q_o}{K_0} \left( \frac{x^2}{L} - L \right) \left( \sin \left( \left( \frac{1}{2} + m \right) \frac{\pi x}{L} \right) \right)_0^L - \int_0^L \frac{2x}{L} \sin \left( \left( \frac{1}{2} + m \right) \frac{\pi x}{L} \right) dx \]

\[ = \frac{Q_o}{K_0} \left( \frac{x^2}{L} - L \right) \left( \left( \frac{2x}{L} \cos \left( \left( \frac{1}{2} + m \right) \frac{\pi x}{L} \right) \right)_0^L - \int_0^L \frac{2x}{L} \cos \left( \left( \frac{1}{2} + m \right) \frac{\pi x}{L} \right) dx \right) \]

\[ = \frac{Q_o}{K_0} \left( \frac{x^2}{L} - L \right) \left( \left( \frac{2x}{L} \cos \left( \left( \frac{1}{2} + m \right) \frac{\pi x}{L} \right) \right)_0^L - \int_0^L \frac{2x}{L} \cos \left( \left( \frac{1}{2} + m \right) \frac{\pi x}{L} \right) dx \right) \]

\[ = - \frac{Q_o}{K_0} \left( \frac{x^2}{L} - L \right) \left( \frac{2x}{L} \sin \left( \left( \frac{1}{2} + m \right) \frac{\pi x}{L} \right) \right)_0^L + \int_0^L \frac{2x}{L} \sin \left( \left( \frac{1}{2} + m \right) \frac{\pi x}{L} \right) dx \]

\[ = - \frac{2}{K_0} \frac{Q_o}{(\frac{1}{2} + m)^3} \left( \frac{(-1)^n}{L^3} \right) \]