BENG 221: Mathematical Methods in Bioengineering

Lecture 8

Solution to the Inhomogeneous Heat Equation with Fourier Series Eigenmode Expansions

References

Haberman APDE, Ch. 2.
Haberman APDE, Ch. 3.

SOLUTION TO INHOMOGENEOUS PDEs

→ Inhomogeneous PDE : non-zero source or sink $f(x,t)$

and/or

Inhomogeneous B.C. : non-zero boundary values and/or flux $u_0(t)$, $u_L(t)$

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(x,t) \quad \text{with} \quad \begin{cases} u(x,0) = g(x) & \text{I.C.} \\ u(0,t) = u_0(t) & \text{B.C.@0} \\ u(L,t) = u_L(t) & \text{B.C.@L} \end{cases}$$

- General case : → Green's functions

- Special case : STATIONARY SOURCE AND B.C. :

  - $f(x,t) = f(x)$ function of x only

  - $\left\{ \begin{aligned} u_0(t) &= u_0 \\ u_L(t) &= u_L \end{aligned} \right.$ constants

→ "POISON TOOTH EXTRACTION" method

Example : NON-HOMOGENEOUS HEAT EQUATION AND/OR B.C. :

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + \frac{1}{c\rho} Q(x) \quad \text{with} \quad \begin{cases} u(x,0) = g(x) & \text{I.C.} \\ u(0,t) = T_0 & \text{B.C@0} \\ u(L,t) = T_L & \text{B.C@L} \end{cases}$$

$\frac{1}{c\rho} Q(x)$ → STATIONARY HEAT SOURCE (independent of time)

T_0, T_L → STATIONARY BOUNDARY VALUES (independent of time)

$$\Rightarrow \quad u(x,t) = u_p(x,t) + u_H(x,t)$$

$u_p(x,t)$ → ONE (ANY!) PARTICULAR SOLUTION TO PDE SATISFYING THE B.C.

$u_H(x,t)$ → EIGENMODE DECOMPOSITION SOLUTION TO THE HOMOGENEOUS PDE WITH HOMOGENEOUS B.C.

"POISON TOOTH" METHOD : If we know a particular solution satisfying the non-homogenous source term and/or non-homogeneous B.C., EXTRACT it to reduce the problem to easier homogeneous form.

Indeed: If $u_p(x,t)$ is a solution, then for $u(x,t)$ to be a solution, its difference $u_H(x,t) = u(x,t) - u_p(x,t)$ must be a solution to the homogeneous PDE with homogeneous B.C.:

$$(+) \quad \frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + \frac{1}{c\rho} Q \quad \text{with} \quad u(0,t) = T_0 \text{ and } u(L,t) = T_L$$

$$(-) \quad \frac{\partial u_p}{\partial t} = D \frac{\partial^2 u_p}{\partial x^2} + \frac{1}{c\rho} Q \quad \text{with} \quad u_p(0,t) = T_0 \text{ and } u_p(L,t) = T_L$$

$$(=) \quad \frac{\partial u_H}{\partial t} = \underbrace{D \frac{\partial^2 u_H}{\partial x^2}}_{\text{HOMOGENEOUS PDE}} \quad \text{with} \quad \underbrace{u_H(0,t) = 0 \text{ and } u_H(L,t) = 0}_{\text{HOMOGENEOUS B.C.}}$$

① Particular solution: STEADY STATE

$$t \to \infty \quad ; \quad \frac{\partial}{\partial t} = 0$$

Always works when the source term and B.C. are independent of time

$\Rightarrow$ $u_p = u_p(x)$ function of x only

$$\frac{\partial^2}{\partial x^2} \to \frac{d^2}{dx^2}$$ PDE becomes an ODE

$$D \frac{d^2 u_p}{dx^2} + \frac{1}{c\rho} Q(x) = 0 \quad \text{with} \begin{cases} u_p(0) = T_0 & \text{B.C.@0} \\ u_p(L) = T_L & \text{B.C@L} \end{cases}$$

or
$$\frac{d^2 u_p}{dx^2} = -\frac{1}{K_0} Q(x) \qquad \left(D = \frac{K_0}{c\rho}\right)$$

$$\Rightarrow \quad u_p(x) = -\frac{1}{K_0} \int_0^x \int_0^{x_1} Q(x_0)\, dx_0\, dx_1 + cx + d$$

(integrate twice)

e.g.: Q = constant (uniform)

$$\Rightarrow \quad u_p(x) = -\frac{Q}{2K_0} x^2 + cx + d$$

B.C.: @ $x=0$: $\quad u_p(0) = T_0 \Rightarrow d = T_0$

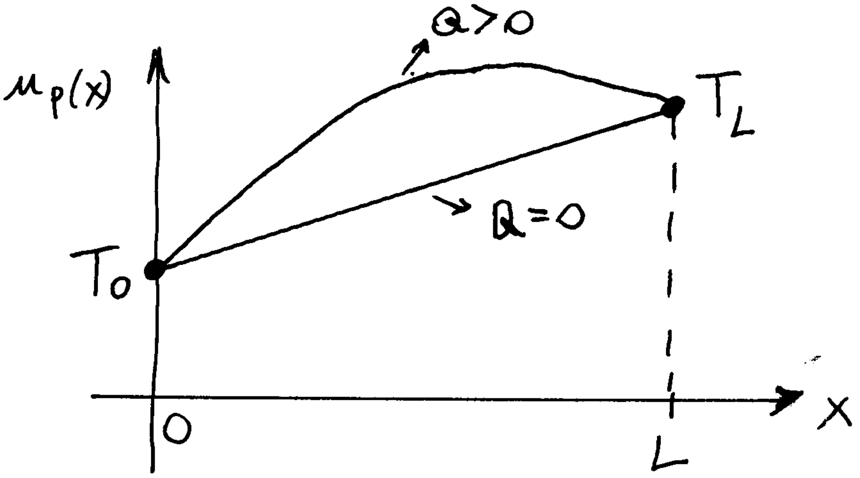
@ $x=L$: $\quad u_p(L) = T_L$

$$\Rightarrow \quad -\frac{Q}{2K_0} L^2 + cL + T_0 = T_L$$

$$\Rightarrow \quad c = \frac{T_L - T_0}{L} + \frac{Q}{2K_0} L$$

$$\Rightarrow \quad u_p(x) = \underbrace{\frac{Q}{2K_0} x(L-x)}_{\text{HEAT SOURCE EFFECT}} + \underbrace{T_0 + (T_L - T_0)\frac{x}{L}}_{\text{BOUNDARY EFFECT}}$$

$$u_p(x) = \frac{Q}{2K_0} x(L-x) + T_0 + (T_L - T_0)\frac{x}{L}$$



(2) Homogeneous solution:

$$\frac{\partial u_H}{\partial t} = D \frac{\partial^2 u_H}{\partial x^2} \quad \text{with} \quad \begin{cases} u_H(0,t) = 0 & \text{HOM. B.C @ 0} \\ u_H(L,t) = 0 & \text{HOM. B.C. @ L} \end{cases}$$

Same old!

$$u_H(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-D\left(\frac{n\pi}{L}\right)^2 t}$$

EIGENMODE EXPANSION

③ Full solution:

$$u(x,t) = u_p(x) + u_H(x,t)$$

$$= u_p(x) + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-D\left(\frac{n\pi}{L}\right)^2 t}$$

STEADY STATE

TRANSIENT

I.C.: $g(x) = u(x,0) = u_p(x) + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$

$$\Rightarrow a_n = \frac{2}{L}\int_0^L \left(g(x_0) - u_p(x_0)\right) \sin\left(\frac{n\pi x_0}{L}\right) dx_0$$

Fourier series

where $u_p(x) = \frac{Q}{2K_0} x(L-x) + T_0 + (T_L - T_0)\frac{x}{L}$

→ Solve the integrals for a_n by integration by parts.

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Lecture 6

Analytic solution of the diffusion/heat equation

The partial differential equation that governs diffusion processes can be solved analytically. If the boundary conditions are simple we can compute this solution readily. The method requires the use of some fundamental mathematical properties.

Linearity

A **linear operator** *L* has the property:

$$L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2)$$

$$\frac{\partial}{\partial t} \quad and \quad \frac{\partial^2}{\partial x^2}$$

are linear operators, therefore the "diffusion operator" $\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$ is a linear operator.

A **linear equation** for $u = u(x,t)$ has the form:

$L(u) = f(x,t)$

$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = f(x)$ is a linear operator on $u(x,t)$ since:

$$\frac{\partial(c_1 u_1 + c_2 u_2)}{\partial t} - k\frac{\partial^2 (c_1 u_1 + c_2 u_2)}{\partial x^2} = f(x,t) =$$
$$c_1\frac{\partial u_1}{\partial t} + c_2\frac{\partial u_2}{\partial t} - k\left(c_1\frac{\partial^2 u_1}{\partial x^2} + c_2\frac{\partial^2 u_2}{\partial x^2}\right) = f(x,t)$$

If $f(x,t) = 0$ then $L(u) = 0$ is a **linear homogeneous** equation.

A fundamental principle:

If u_1 and u_2 satisfy a linear homogenous equation then any arbitrary linear combination $c_1u_1 + c_2u_2$ satisfies the same linear homogenous equation.

The concept of linearity also applies to boundary conditions.

The complete diffusion equation is:

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} + G(x,t)$$

where *G(x,t)* is the material generated or consume within the control volume. The diffusion equation is linear and homogeneous if *G(x,t)* = 0.

Solution of the diffusion equation with specified concentration at the ends of the region, and no sources. Separation of variables.

We propose to solve the linear homogeneous diffusion equation:

$$\frac{\partial u(x,t)}{\partial t} - k\frac{\partial^2 u(x,t)}{\partial x^2} = 0 \quad B.C.s \quad u(0,t) = u(L,t) = 0 \quad I.C. \quad u(x,0) = f(x) \tag{11}$$

in terms of solutions of the form:

$$u(x,t) = w(x)g(t) \tag{12}$$

which reduces PDEs to ODEs.

Accordingly, and to obtain terms that are present in the diffusion equation we take the partial derivatives of (12) and obtain:

$$\frac{\partial u(x,t)}{\partial t} = w(x)\frac{dg(t)}{dt} \qquad \text{and} \qquad \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{d^2 w(x)}{dx^2}g(t)$$

Leading to a new form of the diffusion equation which can now be written as an ordinary differential equation:

$$w(x)\frac{dg(t)}{dt} - kg(t)\frac{d^2 w(x)}{dx^2} = 0 \text{ which can be also written as: } \frac{1}{kg(t)}\frac{dg(t)}{dt} = \frac{1}{w(x)}\frac{d^2 w(x)}{dx^2}$$

where each side is the function of only one variable. One side of this equation, a function of *t*, is equal to function of *x*. This is only possible if both sides equal a constant, which for convenience is taken as $-\lambda$ or:

$$\frac{1}{kg(t)}\frac{dg(t)}{dt} = \frac{1}{w(x)}\frac{d^2 w(x)}{dx^2} = -\lambda \tag{13}$$

which results into two ODEs, namely:

$$\frac{dg}{dt} = -\lambda kg \qquad and \qquad \frac{d^2 w}{dx^2} = -\lambda w$$

General solution of the time dependent equation:

$$g(t) = ce^{-\lambda kt}$$

where c is an arbitrary constant. Note that if $\lambda > 0$ the solution decays in time, which we expect.

Boundary value problems and solution of the *x* dependent equation

Note that $w(x) = 0$ satisfies (11). However this is a trivial solution. Other non trivial solutions that also satisfy the B.C.s exist for specific values of λ. These are called eigenvalues, and a non trivial *w(x)* that exists only for the eigenvalues is called an eigenfunction.

The second order O.D.E. $\frac{d^2w}{dx^2} = -\lambda w$ solution is obtained by substituting $w(x) = e^{rx}$ and obtaining the characteristic polynomial $r^2 = -\lambda$ which yields 3 different types of solutions:

1. $\lambda > 0$ roots are imaginary and complex conjugates, $r = \pm i\sqrt{\lambda}$.
2. $\lambda = 0$, $r = 0,0$.
3. $\lambda < 0$, roots a real, $r = \pm\sqrt{-\lambda}$.

Solutions for $\lambda > 0$.

For $\lambda > 0$ the exponential solutions have imaginary exponents, namely $e^{\pm i\sqrt{\lambda}x}$. Linear combinations of these exponents yield trigonometric function, namely:

$$\frac{1}{2}(e^{i\sqrt{\lambda}x} + e^{-i\sqrt{\lambda}x}) = \cos\sqrt{\lambda}x \quad \textit{and} \quad \frac{1}{2i}(e^{i\sqrt{\lambda}x} - e^{-i\sqrt{\lambda}x}) = \sin\sqrt{\lambda}x$$

Therefore $w(x) = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x$ is a solution since it is a linear combination of solutions.

The B.C. $w(0) = 0$ implies that $c_1 = 0$ and the cosine term vanishes. In order to satisfy the other boundary condition $w(L) = 0$ either $c_2 = 0$ in which case $w(x) = 0$ is a trivial solution, or we search for values of λ that satisfy the relationship $\sin\sqrt{\lambda}L = 0$, namely the zeros of the sine function which occur at $\sqrt{\lambda}L = n\pi$. Therefore

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, 3, \ldots$$

are the eigenvalues λ and the corresponding eigenfunctions are:

$$w(x) = c_2 \sin\sqrt{\lambda}x = c_2 \sin\left(\frac{n\pi x}{L}\right)$$

<u>Is $\lambda = 0$ a non trivial solution</u>?

The equation being solved is:

$$\frac{d^2 w(x)}{dx^2} = -\lambda w(x) = 0$$

Which this leads to $w(x) = c_1 + c_2 x$ and in order to satisfy the boundary conditions *w(x)* = 0 at *x* = *L* then $c_1 = c_2 = 0$, a trivial solution, and the only one possible.

<u>Solutions for λ < 0</u>.

When λ < 0 the roots of the characteristic are $r = \pm\sqrt{-\lambda}$ which is a real number. To avoid confusion it is convenient to set $\lambda = -s$ which results in two independent solutions $e^{\sqrt{s}x}$ and $e^{-\sqrt{s}x}$ leading to the general solution:

$$w(x) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}$$

which can be expressed as hyperbolic functions, since these are linear combinations of exponentials functions, namely

$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right) \quad \sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$

leading to the solution:

$$w(x) = c_3 \cosh\sqrt{s}x + c_4 \sinh\sqrt{s}x$$

Which has a solution that satisfies the boundary conditions only if $c_3 = c_4 = 0$ or $w(x)$ = 0 a trivial solution since $\sinh\sqrt{s}L$ is never zero for a positive argument.

Product solution of the P.D.E with specified boundary conditions

The product solution of (11) therefore exists for λ > 0 and is of the form:

$$u(x,t) = A_n \sin\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t} \qquad n = 1, 2, 3, \ldots$$

Note that as there is a different solution for each *n*, and that as *t* increases all special solutions exponentially decay in time. Furthermore $u(x,t)$ satisfies the special initial condition

$$u(x,0) = A_n \sin\frac{n\pi x}{L}$$

and in general, given the principle of superposition:

$$u(x,t) = \sum_{n=1}^{M} A_n \sin\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t} \tag{14}$$

It should be noted that (14) should satisfy the initial boundary condition $u(x,0) = f(x)$ which requires finding a way to link A_n and $f(x)$ for $t = 0$.

This can be done by noting that the eigenfunctions $\sin\frac{n\pi x}{L}$ satisfy the integral property:

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = 0(m \neq n);\ = \frac{L}{2}(m = n)$$

In other words:

$$\int_0^L \sin^2\frac{n\pi x}{L}\,dx = \frac{L}{2}$$

Note that this integral is associated with calculation of average power in periodic phenomena. In general the average power is calculated by taking the integral

$$\int_0^{2\pi/\omega} \sin^2\omega t\,dt = \left[\frac{1}{2}t - \frac{1}{4}\sin 2\omega t\right]_0^{2\pi/\omega} = \frac{\pi}{\omega}$$

Therefore in (14), for $t = 0$ when $u(x,0) = f(x)$ we can set up the following relationship:

$$\int_0^L f(x)\sin\frac{m\pi x}{L}\,dx = \sum_{n=1}^{\infty} A_n \int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx$$

and whenever $m = n$

$$\int_0^L f(x)\sin\frac{m\pi x}{L}dx = \sum_{n=1}^{\infty} A_m \int_0^L \sin^2\frac{m\pi x}{L}dx$$

and since

$$\int_0^L \sin^2\frac{m\pi x}{L}dx = \frac{L}{2}$$

$$A_m = \frac{2}{L}\int_0^L f(x)\sin\frac{m\pi x}{L}dx$$

A vector view of the computation of *A_n*

Consider the sample problem we just worked with the same boundary conditions. Consider now an ordinary vector, **F**, with components in three dimensional space with coordinates *x, y*, and *z*. The vector is defined by a basis set of the three perpendicular (or orthogonal) unit vectors, $\mathbf{e_x}$, $\mathbf{e_y}$ and $\mathbf{e_z}$, pointing in the three dimensions. To describe the vector **F**, each unit vector has some magnitude, *A*, so that:

$$\mathbf{F} = A_x\mathbf{e_x} + A_y\mathbf{e_y} + A_z\mathbf{e_z}$$

If **F** is known we can find the A_i coefficients by taking the inner (or dot) product of the whole expression with each of the unit vectors in turn. As an example:

$$A_x = \mathbf{F}\cdot\mathbf{e_x} \qquad 15$$

Now consider the expression:

$$A_1\sin(\pi x) + A_2\sin(2\pi x) + A_3\sin(3\pi x) + ... = f(x)$$

We can assume that each function sin(*nπx*) is a basis vector. We can define:

$$\int_0^1 f(x)\sin(n2\pi x)dx \quad = \quad \text{inner product of } f(x) \text{ with n}^{th} \text{ basis vector } \sin(n\pi x)$$

which satisfies the same conditions as our usual operation with three dimensional basis vectors, where we get zero for the inner product of any combination of basis functions, except when we take the inner product with itself. For example:

$$2\int_0^1 \sin(\pi x)\sin(2\pi x)dx = 0$$

$$2\int_0^1 \sin(\pi x)\sin(\pi x)dx = 1$$

This expression is analogous to $\mathbf{e_x} \cdot \mathbf{e_y} = 0$ and $\mathbf{e_x} \cdot \mathbf{e_x} = 1$.

Following this analogy to find the coefficients in front of each basis function we take the inner product of the whole expression with the first basis function (n = 1) as in equation 15:

$$\int_0^1 [A_1 \sin(\pi x) + A_2 \sin(2\pi x) + A_3 \sin(3\pi x) + ... = f(x)]\sin(\pi x)dx$$

Since the basis functions are orthogonal, the inner product is zero for any combination of unlike functions. Therefore the only non zero terms are:

$$\int_0^1 [A_1 \sin(\pi x)\sin(\pi x)dx = \frac{A_1}{2} = \int_0^1 f(x)\sin(\pi x)dx$$

an analogous expression to 15, and for any coefficient:

$$A_n = 2\int_0^1 f(x)\sin(n\pi x)dx$$

The term on the right is the inner product of the first basis function with the known function $f(x)$. Furthermore, just like $\mathbf{F} \cdot \mathbf{e}_x$ gives the projection of **F** in the x direction, $\int_0^1 f(x)\sin(n\pi x)dx$ gives the projection of $f(x)$ on $\sin(n\pi x)$.