BENG 221: Mathematical Methods in Bioengineering

Lecture 7

Heat Conduction

References

Haberman APDE, Ch. 1.

http://en.wikipedia.org/wiki/Heat_equation

HEAT CONDUCTION AND DIFFUSION (IN 1-D)

Haberman Sec. 1.2

$e(x,t)$: thermal energy density $\left[\frac{J}{m^3}\right]$

$\phi(x,t)$: heat flux : flow of thermal energy per unit time per unit surface area $\left[\frac{J}{s\,m^2}\right]$

$Q(x,t)$: heat generation: heat energy generated per unit volume per unit time $\left[\frac{J}{s\,m^3}\right]$

$u(x,t)$: temperature $[K]$ or $[°C]$

Note the analogy with the electrical model (Week 2):

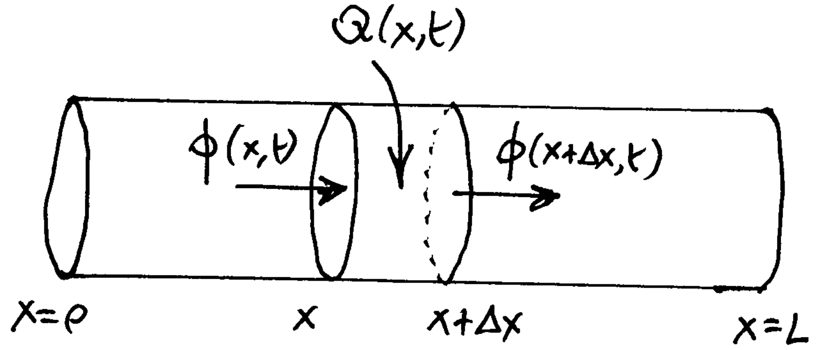
$e \longleftrightarrow q$: charge density $\left[\frac{C}{m^3}\right]$

$\phi \longleftrightarrow i$: current density along the cable $\left[\frac{A}{m^2}\right] = \left[\frac{C}{s\,m^2}\right]$

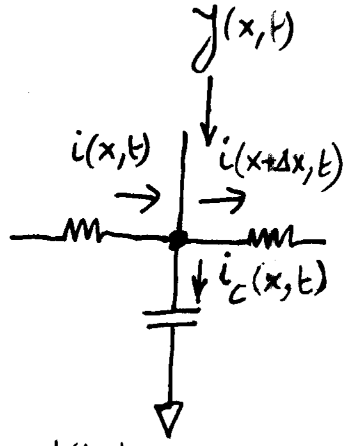
$Q \longleftrightarrow j$: current density through the cable membrane (or generated inside) $\left[\frac{A}{m^3}\right] = \left[\frac{C}{s\,m^3}\right]$

$u \longleftrightarrow v$: voltage $[V]$

- Conservation of heat energy :



or :



Kirchoff's Current Law

$$\frac{\partial}{\partial t}\left(e(x,t) \cdot \Delta x A \right) = \phi(x,t) \cdot A - \phi(x+\Delta x,t) A + Q(x,t)\, \Delta x A$$

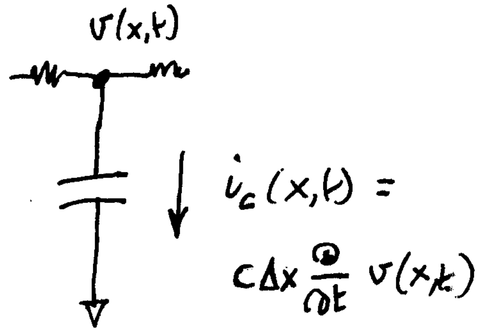
rate of change in heat energy over time = net flux flowing across the boundaries per unit time + heat energy generated inside per unit time

$$\Rightarrow \quad \frac{\partial}{\partial t} e(x,t) = \lim_{\Delta x \to 0} \frac{\phi(x,t) - \phi(x+\Delta x,t)}{\Delta x} + Q(x,t)$$

or $$\frac{\partial e}{\partial t} = - \frac{\partial \phi}{\partial x} + Q$$

$$\left[\frac{1}{s}\right]\left[\frac{J}{m^3}\right] \qquad \left[\frac{1}{m}\right]\left[\frac{J}{s\,m^2}\right] \qquad \left[\frac{J}{s\,m^3}\right] \quad \text{ok} \checkmark$$

- Heat capacity (specific heat):



$$e(x,t) = c(x) \cdot \rho(x) \cdot u(x,t)$$

thermal energy density	specific heat (heat capacity)	mass density	temperature
$\left[\frac{J}{m^3}\right]$	$\left[\frac{J}{kg\,K}\right]$	$\left[\frac{kg}{m^3}\right]$	$[K]$

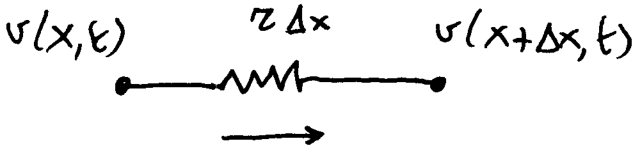
$\downarrow$

heat (thermal energy) required
to raise a unit of mass
by one unit of temperature

(similar to electrical capacitance, except per unit mass rather than volume; hence the $\rho(x)$ multiplicative factor)

$$\Rightarrow \quad c(x) \cdot \rho(x) \cdot \frac{\partial}{\partial t} u(x,t) = -\frac{\partial}{\partial x} \phi(x,t) + Q(x,t)$$

• Heat conduction :



(reciprocal of heat resistance)

$$i(x,t) = \frac{1}{r\Delta x}\cdot\left(v(x,t) - v(x+\Delta x,t)\right) \underset{\Delta x \to 0}{\overset{}{\underset{\lim}{=}}} -\frac{1}{r}\cdot\frac{\partial v}{\partial x}$$

$$\phi = -K_0 \frac{\partial u}{\partial x}$$

Fourier's law of heat conductance
Fick's law of diffusion
Ohm's law

- ϕ: heat flux $\left[\frac{J}{s\,m^2}\right]$
- K_0: thermal conductivity $\left[\frac{J}{s\,m\,K}\right]$
- $\frac{\partial u}{\partial x}$: temperature gradient (field) $\left[\frac{K}{m}\right]$

$$\Rightarrow \quad c(x)\,\rho(x)\,\frac{\partial}{\partial t}u(x,t) = \frac{\partial}{\partial x}\left(K_0(x)\,\frac{\partial}{\partial x}u(x,t)\right) + Q(x,t)$$

General heat equation (or diffusion equation)

$$c(x)\,\rho(x)\,\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0(x)\,\frac{\partial u}{\partial x} \right) + Q(x,t)$$

Special case : $c(x)$, $\rho(x)$, and $K_0(x)$ are uniform constant

$$\Rightarrow \quad \frac{\partial u}{\partial t} = D\,\frac{\partial^2 u}{\partial x^2} + \frac{Q(x,t)}{c\rho}$$

with $D = \dfrac{K_0}{c\rho}$ THERMAL DIFFUSIVITY

units : $\dfrac{K_0}{c\rho}$: $\dfrac{\left[\frac{\cancel{J}}{s\,m\,\cancel{K}}\right]}{\left[\frac{\cancel{J}}{\cancel{kg}\,\cancel{K}}\right]\left[\frac{\cancel{kg}}{m^3}\right]} = \left[\dfrac{m^2}{s}\right]$ ok ✓

- A general, non-zero $Q(x,t)$ and/or B.C. make the system inhomogeneous : a linear combination of valid solutions is not a valid solution
- However, the solution to the inhomogeneous PDE with inhomogeneous BC can be written as a sum of the solution to the homogeneous PDE with homogeneous BC, plus a PARTICULAR inhomogeneous solution. (LECTURE 6)

Lecture 5
BENG 221
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Heat conduction

<u>Conduction of heat (thermal energy) in a rod</u>.

Heat flow: transfer of thermal energy. Thermal energy is the consequence of the motion of molecules. This occurs as a consequence of two processes: 1) kinetic energy of vibration is transmitted to neighboring molecules due to collisions. Therefore thermal energy spreads even though the molecules do not change location. In this situation thermal energy "diffuses". Vibrating molecules move taking energy with them: this is called "convection".

Conduction in an insulated rod oriented in the x direction, constant cross-section A, boundaries $x = 0$, $x = L$. The state of the rod is described by its temperature $u(x,t)$ which characterizes the thermal energy in the rod. The thermal energy in the rod is related to the temperature by the specific heat:

$c(x,t)$ = energy that must be supplied to a unit of mass to raise temperature one unit.

The thermal energy per unit mass or energy density $e(x,t) = c(x,t)\, u(x,t)$

The thermal energy in per unit volume is:

$E(x,t) = \rho(x)\, e(x,t) = \rho(x)\, c(x,t)\, u(x,t)$ where $\rho(x)$ is the density (mass per unit volume) (1)

And for constant $\rho(x)$ and $c(x,t)$:

$E(x,t) = \rho\, c\, u(x,t)$

<u>The heat equation</u>

The rate at which $E(x,t)$ changes in time in a volume $\Delta V = A\Delta x$ of the rod bound at x and $x + \Delta x$ at time t is equal to:

$$\left.\frac{\partial E(x,t)}{\partial t}\right|_{\Delta Vol} = \rho c \frac{\partial e(x,t)}{\partial t} A\Delta x \qquad (2)$$

The rate of change of thermal energy (2) is equal to the difference between the rate of influx of thermal energy $\Phi(x,t)$, at a cross section A at x and the rate at which thermal energy exits through the cross section A at $x+\Delta x$. Therefore the rate of change of termal energy in the control volume is the difference between fluxes $\Phi(x,t)$ at x and $x+\Delta x$ namely that enter and leave the volume $dV = A\Delta x$. For constant $\rho(x)\, c(x,t)$:

$$\Phi(x,t)A \tag{3}$$

and at $x + \Delta x$:

$$\Phi(x + \Delta x, t)A \tag{4}$$

The difference between (3) and (4) is the rate of thermal energy production (or consumption) *G(x,t)* per unit volume generated or absorbed in the volume $\Delta V = A\Delta x$:

$$G(x,t)\ A\ \Delta x \tag{5}$$

The difference between (3) and (4) plus 5 must equal (2) since energy is conserved, and taking the limit:

$$\lim_{\Delta x \to 0} \frac{\Phi(x,t) - \Phi(x + \Delta x, t)}{\Delta x} + G(x,t) = \rho c \frac{\partial u(x,t)}{\partial t}$$

Therefore:

$$\rho c \frac{\partial u(x,t)}{\partial t} = -\frac{\partial \Phi(x,t)}{\partial x} + G(x,t) \tag{6}$$

The rate at which thermal energy is transferred is found experimentally to be a function of the negative temperature gradient according to:

$$\Phi(x,t) = -K_0 \frac{\partial u(x,t)}{\partial x} \tag{7}$$

Which is the first law of diffusion. Note that the constant K_0 introduces the time dependence in this expression. Introducing (7) into (6) yields:

$$\frac{\partial u(x,t)}{\partial t} = k \frac{\partial^2 u(x,t)}{\partial x^2} + Q(x,t) \qquad k = \frac{K_0}{c\rho}$$

(the second law of diffusion) where *k* is the thermal diffusivity, playing a similar role as the diffusion constant in the diffusion equation.

<u>Exact derivation of the heat equation</u>.

Consider a finite segment along the rod with boundaries at $x = a$ and $x = b$. The total heat energy in the segment with cross sectional area $A = 1$ is:

$$Total\ thermal\ energy = \int_a^b E(x,t)\,dx$$

The total thermal energy it the volume bound by x = a and x = b changes as a consequence of the flux of energy at the boundaries and the energy generated or absorbed within, namely:

$$\frac{d}{dt}\int_a^b E(x,t)dt = \Phi(a,t) - \Phi(b,t) + \int_a^b G(x,t)\,dx = \int_a^b \frac{\partial}{\partial t} E(x,t)\,dx \qquad (8)$$

note that:

$$\Phi(a,t) - \Phi(b,t) = -\int_a^b \frac{\partial}{\partial x}\Phi(x,t)dx$$

(a fundamental calculus theorem) therefore in (8):

$$\int_a^b \left(\frac{\partial}{\partial t}E(x,t) + \frac{\partial}{\partial x}\Phi(x,t) - G(x,t)\right)dx = 0 \qquad (9)$$

or

$\frac{\partial E}{\partial t} = -\frac{\partial \Phi}{\partial x} + G$ and from (7) $\frac{\partial E}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + G$

and from (2) and $E(x,t) = \rho\, c\, u(x,t)$:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \frac{1}{\rho c}G(t) = k\frac{\partial^2 u}{\partial x^2} + Q(t)$$

<u>Boundary conditions</u>

Temperature is specified at given locations for all times:

$$u(0,t) = u_A(t) = T_A \qquad u(L,t) = u_B(t) = T_B$$

Specification of flux. If the flux of thermal energy is specified at a cross section at $x = L$, then:

$$K_0 \frac{\partial u(L,t)}{\partial x} = \Phi_L(L,t)$$

If there is no flux of thermal energy then the boundary is insulated:

$$\frac{\partial u(L,t)}{\partial x} = 0$$

Newton law of cooling: the heat flux out of the rod is proportional to the surrounding temperature $u_B(t)$:

$$-K_0 \frac{\partial u(0,t)}{\partial x} = -H\left[u(0,L,t) - u_B(t)\right]$$

Radiative heat loss:

$$-K_0 \frac{\partial u(0,t)}{\partial x} = R\left[u^4(0,L,t) - u_B^4(t)\right]$$

<u>Solutions for a uniform rod with insulated sides.</u>

<u>Equilibrium temperature distribution. Specified temperature</u>.

If there are no sources of heat, and the boundary conditions are independent of time then the partial differential heat equation reduces to an ordinary differential equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 u}{dx^2} = 0 \quad B.C. \quad u(0,0) = T_a \quad u(L,0) = T_b \tag{10}$$

the solution is:

$$u(x) = T_a + \frac{T_b - T_a}{L} x \tag{11}$$

<u>Velocity of the motion of thermal energy and temperature</u>.

Thermal energy per unit volume $E(x,t) = \rho\, c\, u(x,t)$

To describe the flux of thermal energy that crosses a cross section A we define a different type of time derivative which describe the amount of thermal energy $\rho c u(x,t) A \Delta x$ that crosses the surface A in time Δx or:

$$\Phi(x,t) = \lim_{\Delta t \to 0} \frac{\rho\, c\, u(x,t) A \Delta x}{\Delta t} = \rho\, c\, A u(x) \frac{dx}{dt} = \rho\, c\, A u(x) v(x)$$

In the previous case for the time independent conditions with specified temperature boundary conditions the flux is constant (using 11 to define the flux) and if $T_a = 0$ then at $x = 0$ the velocity of heat transfer would be infinite.

Equilibrium temperature distribution. Insulated boundaries.

In this problem a rod of uniform cross-section has an initial temperature distribution $u(x,0) = f(x)$ at $t = 0$. The ends are insulated and no thermal energy leaves the rod. Since no energy leaves the system as a consequence of the flux of thermal energy being zero at $x = 0,L$ the total thermal energy in the system does not change with time, namely:

$$\frac{d}{dt}\int_0^L E(x,t)dx = \Phi(L,t) - \Phi(0,t) = 0 \tag{9}$$

As before:

$$\int_0^L \frac{d}{dt} E(x,t)dx = \Phi(L,t) - \Phi(0,t) = 0 = \int_0^L \frac{d}{dx}\Phi(x,t)dx = -\int_0^L K\frac{\partial^2 u(x,t)}{\partial x^2}\, dx$$

As above, the partial differential heat equation reduces to an ordinary differential equation:

$$\frac{d^2u}{dx^2} = 0 \quad B.C.'s \quad \frac{du(0,t)}{dx} = 0 \quad \frac{du(L,t)}{dx} = 0$$

The solution is: $u(x,t) = c_1 + c_2 x$

since the slope is zero at both ends $c_2 = 0$ and any $u(x,t) = c_1$ is a solution of the problem and satisfies the differential equation and *B.C.*s. Note that there is no unique solution since any constant temperature is a solution to the problem. There is no uniqueness because there is no information about the initial conditions. The unique solution is obtained by taking into consideration the initial conditions of the time dependant problem, when $u(x,0) = f(x)$. From (8) and (6):

$$\frac{d}{dt}\int_0^L E(x,t)dx = \Phi(L,t) - \Phi(0,t) = 0 = \frac{d}{dt}\int_0^L \rho c u(x,t)dx$$

Therefore for the initial condition $u(x,0) = f(x)$ and for any time, since $u(x,t) = c_1$

$$c\rho\int_0^L u(x,t)\,dx = c\rho\int_0^L c_1 dx = constant = \int_0^L c\rho\, f(x)\,dx \quad \& \quad c_1 = \frac{1}{L}\int_0^L f(x)\,dx$$