BENG 221: Mathematical Methods in Bioengineering

Lecture 6

Solutions to PDEs over Bounded and Unbounded Domains

References

Haberman APDE, Ch. 2.

Haberman APDE, Ch. 3.

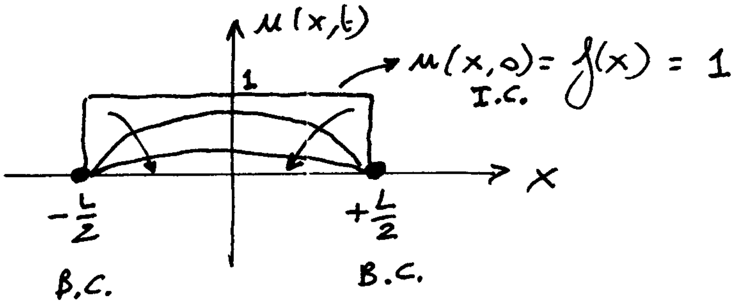
Haberman APDE, Ch. 10.

SOLUTION TO PDEs ON BOUNDED DOMAINS

e.g.: Diffusion equation on interval $\left[-\frac{L}{2}, +\frac{L}{2}\right]$:

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \begin{cases} IC: & u(x,0) = f(x) \\ BC: & \begin{cases} u(-\frac{L}{2}, t) = 0 \\ u(+\frac{L}{2}, t) = 0 \end{cases} \end{cases}$$

e.g.:



Try solution by separation of variables:

$$u(x,t) = \phi(x) \cdot G(t)$$

$$\Rightarrow \quad \phi(x) \cdot \frac{dG(t)}{dt} = D \cdot \frac{d^2\phi(x)}{dx^2} \cdot G(t)$$

$$\Rightarrow \quad \underbrace{\frac{\frac{d^2\phi(x)}{dx^2}}{\phi(x)}}_{\text{function of } x \text{ ONLY}} = \underbrace{\frac{1}{D} \frac{\frac{dG(t)}{dt}}{G(t)}}_{\text{function of } t \text{ ONLY}} = -\lambda$$

$\Rightarrow$ must equal a constant in x AND t

- Time-dependent part:

$$\frac{dG}{dt} = -\lambda D\, G \implies G(t) = \underbrace{G(0)}_{\substack{\neq 0 \\ \text{otherwise: trivial} \\ \text{zero solution}}} \cdot e^{-\lambda D t} \neq 0 \quad \forall t$$

- Space-dependent part:

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad \text{with B.C.} \begin{cases} u(-\frac{L}{2}, t) = \phi(-\frac{L}{2}) \cdot G(t) = 0 & \forall t \\ u(+\frac{L}{2}, t) = \phi(+\frac{L}{2}) \cdot G(t) = 0 & \forall t \end{cases}$$

$$\implies \phi(-\tfrac{L}{2}) = \phi(\tfrac{L}{2}) = 0$$

- $\lambda = 0$? $\implies \phi(x) = Ax + B \underset{\text{B.C.}}{\implies} A = B = 0$ trivial zero solution

- $\lambda < 0$? $\implies \phi(x) = A\underbrace{e^{\sqrt{-\lambda}x}}_{\neq 0} + B\underbrace{e^{-\sqrt{-\lambda}x}}_{\neq 0} \underset{\text{B.C.}}{\implies} A = B = 0$ also trivial zero solution

- $\lambda > 0$! $\implies \phi(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$

$$\text{B.C.}: \begin{cases} A\cos(\sqrt{\lambda}\frac{L}{2}) + B\sin(\sqrt{\lambda}\frac{L}{2}) = 0 \\ A\cos(\sqrt{\lambda}\frac{L}{2}) - B\sin(\sqrt{\lambda}\frac{L}{2}) = 0 \end{cases}$$

$$\implies A\cos(\sqrt{\lambda}\tfrac{L}{2}) = B\sin(\sqrt{\lambda}\tfrac{L}{2}) = 0$$

A and B can't BOTH be zero for a non-trivial solution

Similarly $\cos(\sqrt{\lambda}\frac{L}{2})$ and $\sin(\sqrt{\lambda}\frac{L}{2})$ can't BOTH be zero

$\implies$ either

↗ $B = 0$ AND $\cos(\sqrt{\lambda}\frac{L}{2}) = 0$

$\implies \sqrt{\lambda}\frac{L}{2} = \frac{\pi}{2} + n\pi$, or: $\sqrt{\lambda} = \frac{\pi}{L} + \frac{2n\pi}{L}$

↘ $A = 0$ AND $\sin(\sqrt{\lambda}\frac{L}{2}) = 0$

$\implies \sqrt{\lambda}\frac{L}{2} = n\pi$, or: $\sqrt{\lambda} = \frac{2n\pi}{L}$

Solutions: $u(x,t) = A \cos\left(\frac{(2n+1)\pi x}{L}\right) e^{-D\left(\frac{(2n+1)\pi}{L}\right)^2 t}$ $\quad n = 0, 1, 2, \dots$

(choose $G(0)=1$)

or: $u(x,t) = B \sin\left(\frac{2n\pi x}{L}\right) e^{-D\left(\frac{2n\pi}{L}\right)^2 t}$ $\quad n = 1, 2, 3, \dots$

All these solutions satisfy the PDE and the B.C.

Problem is: none of these satisfy the I.C.!

Superposition to the rescue: try a linear combination of solutions (itself a solution to the PDE and B.C.) to satisfy the I.C.

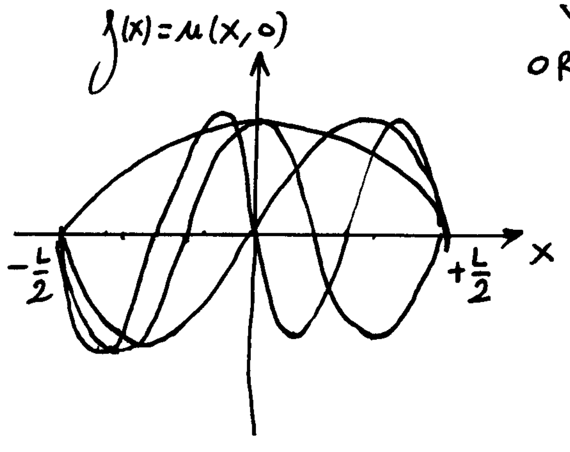
$$\Rightarrow u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{(2n+1)\pi x}{L}\right) e^{-D\left(\frac{(2n+1)\pi}{L}\right)^2 t}$$

"EIGENMODE" EXPANSION

$$+ \sum_{n=1}^{\infty} B_n \sin\left(\frac{2n\pi x}{L}\right) e^{-D\left(\frac{2n\pi}{L}\right)^2 t}$$

I.C.: $\underset{!}{f(x)} = u(x,0) = \sum_{n=0}^{\infty} \underset{?}{A_n} \cos\left(\frac{(2n+1)\pi x}{L}\right) + \sum_{n=1}^{\infty} \underset{?}{B_n} \sin\left(\frac{2n\pi x}{L}\right)$

ORTHOGONAL SET OF BASIS FUNCTIONS



$$f(x) = \sum_{n=0}^{\infty} A_n \phi_n(x) + \sum_{n=1}^{\infty} B_n \phi'_n(x)$$

with
$$\begin{cases} \phi_n(x) = \cos\left(\frac{(2n+1)\pi x}{L}\right) \\ \phi'_n(x) = \sin\left(\frac{2n\pi x}{L}\right) \end{cases}$$

The set $\{\phi_n(x), \phi'_n(x)\}$ is orthogonal in that:

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \phi_n(x)\, \phi_m(x)\, dx = \frac{L}{2}\delta_{nm} \quad \text{where} \quad \delta_{nm} = \begin{cases} 1 & n=m \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \phi'_n(x)\, \phi'_m(x)\, dx = \frac{L}{2}\delta_{nm}$$

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \phi_n(x)\, \phi'_m(x)\, dx = 0$$

$$\Rightarrow \int_{-\frac{L}{2}}^{+\frac{L}{2}} f(x)\, \phi_m(x)\, dx = \sum_{n=0}^{\infty} A_n \frac{L}{2}\delta_{nm} = A_m \frac{L}{2}$$

$$\int_{-\frac{L}{2}}^{+\frac{L}{2}} f(x)\, \phi'_m(x)\, dx = \sum_{n=1}^{\infty} B_n \frac{L}{2}\delta_{nm} = B_m \frac{L}{2}$$

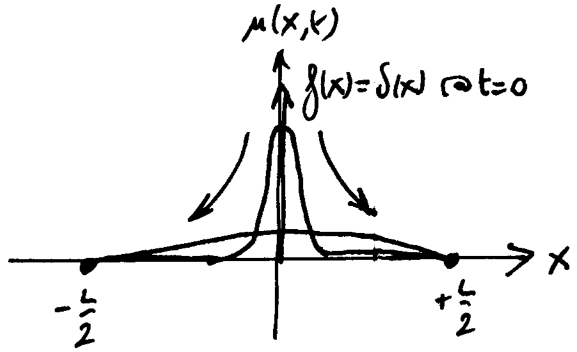
$$\Rightarrow \quad u(x,t) = \sum_{n=0}^{\infty} A_n \phi_n(x)\, e^{-D\lambda_n t} + \sum_{n=1}^{\infty} B_n \phi_n'(x)\, e^{-D\lambda_n' t}$$

where $\lambda_n = \left(\frac{(2n+1)\pi}{L}\right)^2$ $\qquad \lambda_n' = \left(\frac{2n\pi}{L}\right)^2$

$\phi_n(x) = \cos\left(\frac{(2n+1)\pi x}{L}\right)$ $\qquad \phi_n'(x) = \sin\left(\frac{2n\pi x}{L}\right)$

and $A_n = \frac{2}{L}\int_{-\frac{L}{2}}^{+\frac{L}{2}} f(x)\, \phi_n(x)\, dx$ $\qquad B_n = \frac{2}{L}\int_{-\frac{L}{2}}^{+\frac{L}{2}} f(x)\, \phi_n'(x)\, dx$

e.g.: $f(x) = \delta(x)$



$A_n = \frac{2}{L}\phi_n(0) = \frac{2}{L}$ $\qquad B_n = \frac{2}{L}\phi_n'(0) = 0$

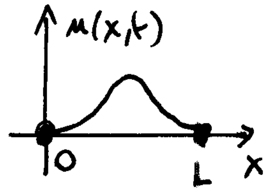
$$\Rightarrow \quad u(x,t) = \frac{2}{L}\sum_{n=0}^{\infty} \cos\left(\frac{(2n+1)\pi x}{L}\right) e^{-D\left(\frac{(2n+1)\pi}{L}\right)^2 t}$$

NOTE: $\lim_{L\to\infty} u(x,t) = \frac{1}{\sqrt{4\pi D t}}\, e^{-\frac{x^2}{4Dt}}$ Gaussian solution to the UNBOUNDED problem

Variations and generalizations:

HOMOGENEOUS PDEs WITH HOMOGENEOUS B.C.

1) Homogeneous diffusion with zero VALUE B.C. :



$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \begin{cases} u(x,0) = g(x) & : \text{I.C. @ } t=0 \\ u(0,t) = 0 & : \text{B.C. @ } x=0 \text{ (ZERO VALUE)} \\ u(L,t) = 0 & : \text{B.C. @ } x=L \text{ (ZERO VALUE)} \end{cases}$$

Separation of variables: $u(x,t) = \phi(x) . G(t)$

As before : $\begin{cases} G(t) = e^{-\lambda D t} \\ \phi(x) = A \cos(\sqrt{\lambda} x) + B \sin(\sqrt{\lambda} x) \end{cases}$

But now with different boundary conditions and resulting eigenvalues:

$$\begin{cases} \text{B.C @ } x=0: & \phi(0) = 0 \Rightarrow A = 0 \\ \text{B.C @ } x=L: & \phi(L) = 0 \Rightarrow \sin(\sqrt{\lambda} L) = 0 \Rightarrow \sqrt{\lambda} = \frac{n\pi}{L} \quad (n=1,2,\dots) \end{cases}$$

Resulting eigenmode expansion :

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L} x\right) e^{-D\left(\frac{n\pi}{L}\right)^2 t}$$

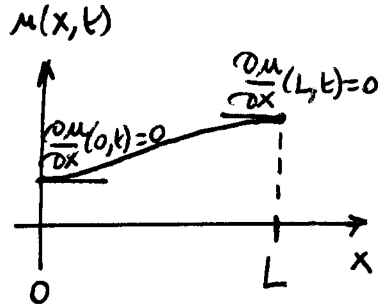
Coefficients B_n in the eigenmode expansion satisfy:

$$\text{I.C. @ } t=0 : \quad u(x,0) = g(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L} x\right)$$

or, by orthogonality of the basis functions over $[0, L]$:

$$B_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx \qquad n = 1, 2, 3, \dots$$

2) Homogeneous diffusion with zero FLUX B.C. :



"Flux" or "Flow" : $-D\dfrac{\partial u}{\partial x}$

(e.g.: "Current" $i(x,t) = -r\dfrac{\partial v}{\partial x}$ in the cable)

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \begin{cases} u(x,0) = g(x) & : \text{ I.C. @ } t=0 \\ \dfrac{\partial u}{\partial x}(0,t) = 0 & : \text{ B.C. @ } x=0 \text{ (ZERO FLUX)} \\ \dfrac{\partial u}{\partial x}(L,t) = 0 & : \text{ B.C @ } x=L \text{ (ZERO FLUX)} \end{cases}$$

Again $u(x,t) = \phi(x) \cdot G(t)$ with $\begin{cases} G(t) = e^{-\lambda D t} \\ \phi(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \end{cases}$

Flux B.C now result in eigenvalues :

$$\begin{cases} \text{B.C @ } x=0 : & \dfrac{d\phi}{dx}(0) = 0 \implies B = 0 \\ \text{B.C @ } x=L : & \dfrac{d\phi}{dx}(L) = 0 \implies \sin(\sqrt{\lambda}L) = 0 \implies \sqrt{\lambda} = \dfrac{n\pi}{L} \quad (n = 0,1,2,\dots) \end{cases}$$

INCLUDE CONSTANT!

and eigenmodes :

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-D\left(\frac{n\pi}{L}\right)^2 t} = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-D\left(\frac{n\pi}{L}\right)^2 t}$$

Coefficients A_n in the eigenmode expansion satisfy :

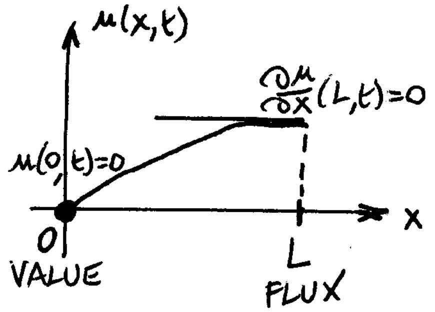
I.C. @ $t=0$: $u(x,0) = g(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$

or, by orthogonality of the basis functions over $[0,L]$:

$$A_0 = \frac{1}{L}\int_0^L g(x)\,dx \qquad \left(\tfrac{1}{L} \text{ because } \int_0^L 1\,dx = L\right)$$

$$A_n = \frac{2}{L}\int_0^L g(x)\cos\left(\frac{n\pi}{L}x\right)dx \qquad n = 1,2,3,\dots$$

3) Homogeneous diffusion with mixed zero VALUE and zero FLUX B.C.: e.g.



$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \begin{cases} u(x,0) = g(x) & : \text{I.C. @ } t=0 \\ u(0,t) = 0 & : \text{B.C. @ } x=0 \text{ (ZERO VALUE)} \\ \frac{\partial u}{\partial x}(L,t) = 0 & : \text{B.C. @ } x=L \text{ (ZERO FLUX)} \end{cases}$$

Again $u(x,t) = \phi(x) \cdot G(t)$ with $\begin{cases} G(t) = e^{-\lambda D t} \\ \phi(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \end{cases}$

Mixed value-flux B.C. now result in eigenvalues:

$$\begin{cases} \text{B.C. @ } x=0 : & \phi(0) = 0 \quad \Rightarrow \quad A = 0 \\ \text{B.C @ } x=L : & \frac{d\phi}{dx}(L) = 0 \quad \Rightarrow \quad \cos(\sqrt{\lambda}L) = 0 \Rightarrow \sqrt{\lambda} = \frac{(2n+1)\pi}{2L} \quad (n=0,1,2\ldots) \end{cases}$$

and eigenmodes:

$$u(x,t) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{(2n+1)\pi}{2L}x\right) e^{-D\left(\frac{(2n+1)\pi}{2L}\right)^2 t}$$

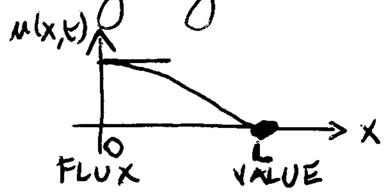
where the coefficients B_n satisfy:

$$\text{I.C. @ } t=0 : \quad u(x,0) = g(x) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{(2n+1)\pi}{2L}x\right)$$

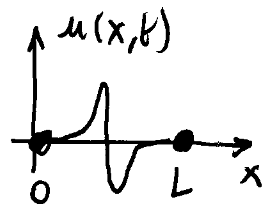
or, again by orthogonality of the basis functions over $[0,L]$:

$$B_n = \frac{2}{L} \int_0^L g(x) \left(\sin \frac{(2n+1)\pi}{2L} x\right) dx \quad , \quad n = 0,1,2,3,\ldots$$

<u>NOTE</u>: Same for flux-value B.C., except replacing "sin" with "cos"



4) Homogeneous wave equation with zero value B.C.:



$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \begin{cases} u(x,0) = g(x) & : \text{VALUE I.C. @ } t=0 \\ \frac{\partial u}{\partial t}(x,0) = h(x) & : \text{VELOCITY I.C. @ } t=0 \\ u(0,t) = 0 & : \text{ZERO VALUE B.C. @ } x=0 \\ u(L,t) = 0 & : \text{ZERO VALUE B.C. @ } x=L \end{cases}$$

Separation of variables : $u(x,t) = \phi(x) \cdot G(t)$

$$\frac{\frac{d^2\phi(x)}{dx^2}}{\phi(x)} = \frac{1}{c^2} \frac{\frac{d^2 G(t)}{dt^2}}{G(t)} = -\lambda$$

- SPACE : $\phi(x)$: $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$, with SAME B.C. as before

$\Rightarrow$ SAME eigenfunctions $\phi(x)$ as for diffusion with same B.C.!!

B.C. : $\phi(x) = \sin(\sqrt{\lambda}\, x)$ with $\sqrt{\lambda} = \frac{n\pi}{L}$, $n = 1, 2, 3, \ldots$

- TIME : $G(t)$: $\frac{d^2 G}{dt^2} + \lambda c^2 G = 0$

$$\Rightarrow G(t) = C\cos(\sqrt{\lambda}\, ct) + D\sin(\sqrt{\lambda}\, ct)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(C_n \cos\left(\frac{n\pi}{L}ct\right) + D_n \sin\left(\frac{n\pi}{L}ct\right)\right)$$

I.C: $$\begin{cases} u(x,0) = g(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right) \overset{\text{ORTH.}}{\Rightarrow} C_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx \\ \frac{\partial u}{\partial t}(x,0) = h(x) = \sum_{n=1}^{\infty} \frac{n\pi}{L} c\, D_n \sin\left(\frac{n\pi x}{L}\right) \overset{\text{ORTH.}}{\Rightarrow} D_n = \frac{2}{n\pi c}\int_0^L h(x)\sin\left(\frac{n\pi}{L}x\right)dx \end{cases}$$

5) Homogeneous wave equation with zero FLUX or MIXED B.C.:

Similar, with SAME $\left\{ \begin{array}{ll} \text{eigenfunctions} & \phi_n(x) \\ \text{eigenvalues} & \lambda_n \end{array} \right\}$ as for diffusion with same B.C.

$$\Rightarrow u(x,t) = \sum_{\substack{n=0 \\ (1)}}^{\infty} \phi_n(x) \left(C_n \cos(\sqrt{\lambda_n}\, ct) + D_n \sin(\sqrt{\lambda_n}\, ct) \right)$$

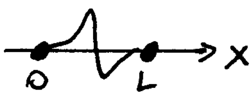
where $\dfrac{d^2\phi_n}{dx^2} + \lambda_n \phi_n = 0$ with $\left\{ \begin{array}{l} \text{ZERO VALUE/FLUX B.C. @ } x=0 \\ \text{ZERO VALUE/FLUX B.C. @ } x=L \end{array} \right.$

identical to the diffusion problem with same B.C.

I.C.: $\left\{ \begin{array}{l} u(x,0) = g(x) = \sum_n C_n \phi_n(x) \overset{\text{ORTH.}}{\Longrightarrow} C_n = \dfrac{\int_0^L g(x)\cdot\phi_n(x)\,dx}{\int_0^L (\phi_n(x))^2\,dx} \\[2ex] \dfrac{\partial u}{\partial t}(x,0) = h(x) = \sum_n \sqrt{\lambda_n}\, c\, D_n \phi_n(x) \overset{\text{ORTH.}}{\Longrightarrow} D_n = \dfrac{\int_0^L h(x)\cdot\phi_n(x)\,dx}{\sqrt{\lambda_n}\, c \int_0^L (\phi_n(x))^2\,dx} \end{array} \right.$

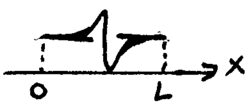
where:

- VALUE-VALUE B.C.: $\phi_n(x) = \sin(\sqrt{\lambda_n}\, x)$ $\qquad \sqrt{\lambda_n} = \dfrac{n\pi}{L}$, $n = 1, 2, 3, \ldots$



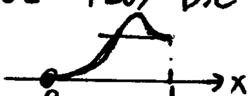
$\int_0^L (\phi_n(x))^2 dx = \dfrac{L}{2}$

- FLUX-FLUX B.C.: $\phi_n(x) = \cos(\sqrt{\lambda_n}\, x)$ $\qquad \sqrt{\lambda_n} = \dfrac{n\pi}{L}$, $n = \underline{0}, 1, 2, 3, \ldots$



$\int_0^L (\phi_n(x))^2 dx = \dfrac{L}{2}$ $n > 0$; $\int_0^L (\phi_0(x))^2 dx = L$

- VALUE - FLUX B.C.: $\phi_n(x) = \sin(\sqrt{\lambda_n}\, x)$ $\qquad \sqrt{\lambda_n} = (n+\tfrac{1}{2})\dfrac{\pi}{L}$, $n = 0, 1, 2, \ldots$



$\int_0^L (\phi_n(x))^2 dx = \dfrac{L}{2}$

- FLUX- VALUE $\phi_n(x) = \cos(\sqrt{\lambda_n}\, x)$ $\qquad$ (SAME)

(SAME)

e.g.:

- Diffusion with VALUE–VALUE B.C & I.C. $g(x) = 1 \quad (0 \le x \le L)$

$$\Rightarrow u(x,t) = \sum_{n \geq 1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-D\left(\frac{n\pi}{L}\right)^2 t}$$

$$\text{with } B_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx \underset{\alpha = \frac{n\pi}{L}x}{=} \frac{2}{n\pi}\int_0^{n\pi} \sin\alpha \, d\alpha = \frac{2}{n\pi}\Big[-\cos\alpha\Big]_0^{n\pi}$$

$$= \begin{cases} \frac{4}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

$$\Rightarrow u(x,t) = \sum_{\substack{n = 1,3,5\dots \\ (\text{ODD})}}^{\infty} \frac{4}{n\pi} \sin\left(\frac{n\pi x}{L}\right) e^{-D\left(\frac{n\pi}{L}\right)^2 t}$$

- Wave with VALUE–FLUX B.C. & I.C. $\begin{cases} g(x) = 0 \\ h(x) = \delta(x-L) \end{cases}$ (kick at the end)

$$\Rightarrow u(x,t) = \sum_{n=0}^{\infty} D_n \sin\left(\left(n+\tfrac{1}{2}\right)\tfrac{\pi}{L}x\right) \sin\left(\left(n+\tfrac{1}{2}\right)\tfrac{\pi}{L}ct\right)$$

$$\text{with } D_n = \frac{2}{\left(n+\frac{1}{2}\right)\pi c}\int_0^L h(x)\sin\left(\left(n+\tfrac{1}{2}\right)\tfrac{\pi}{L}x\right)dx = \frac{2}{\left(n+\frac{1}{2}\right)\pi c}\underbrace{\sin\left(\left(n+\tfrac{1}{2}\right)\pi\right)}_{=(-1)^n}$$

$$\Rightarrow u(x,t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(n+\frac{1}{2}\right)\pi c}\cdot\left(\cos\left(\left(n+\tfrac{1}{2}\right)\tfrac{\pi}{L}(x-ct)\right) - \cos\left(\left(n+\tfrac{1}{2}\right)\tfrac{\pi}{L}(x+ct)\right)\right)$$

$$\sin\alpha \sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$
for all α, β

• Diffusion with PERIODIC B.C.:

$$\frac{\partial u}{\partial t} = D \frac{\partial u^2}{\partial x^2} \quad \text{with} \quad \begin{cases} IC: & u(x,0) = g(x), \text{ periodic w/ period } L \\ BC: & u(x,t) = u(x+L,t), \text{ periodic w/ period } L \end{cases}$$

Separation of variables: $u(x,t) = \phi(x) \cdot G(t)$

As before: $\begin{cases} G(t) = e^{-\lambda D t} \\ \phi(x) = A \cos\sqrt{\lambda}x + B \sin\sqrt{\lambda}x \end{cases}$

But now with different boundary conditions and resulting eigenvalues:

$\phi(x) =$ periodic w/ period L $\Rightarrow$ $\sqrt{\lambda} \cdot L = n \cdot 2\pi$ with $n = 0, 1, 2 \ldots$

Resulting eigenmode expansion:

$$u(x,t) = \sum_{n=0}^{\infty} \left(A_n \cos\left(\frac{n2\pi}{L}x\right) + B_n \sin\left(\frac{n2\pi}{L}x\right) \right) e^{-D\left(\frac{n2\pi}{L}\right)^2 t}$$

Coefficients A_n and B_n satisfy:

$$u(x,0) = \sum_{n=0}^{\infty} \left(A_n \cos\left(\frac{n2\pi}{L}x\right) + B_n \sin\left(\frac{n2\pi}{L}x\right) \right) = g(x)$$

or:

$$A_n = \frac{2}{L} \int_0^L g(x) \cos\left(\frac{n2\pi}{L}x\right) dx$$

$$B_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n2\pi}{L}x\right) dx$$

SOLUTION TO PDEs ON INFINITE DOMAINS

Diffusion without source term (homogeneous PDE):

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \begin{cases} \text{I.C.:} & u(x,0) = g(x), \quad -\infty \leq x \leq +\infty \\ \text{B.C.:} & u(-\infty,t) = 0 \\ & u(+\infty,t) = 0 \end{cases}$$

Solution in the Fourier domain in space (x variable):

Let $U(\xi,t) = \underbrace{\mathcal{F}_x}_{\text{Fourier in } x}\left(u(x,t)\right) = \int_{-\infty}^{+\infty} u(x,t)\, e^{-j\xi x}\, dx$

$$\Rightarrow \quad \mathcal{F}_x\left(\frac{\partial}{\partial x} u(x,t)\right) = -j\xi\, U(\xi,t)$$

$$\mathcal{F}_x\left(\frac{\partial^2}{\partial x^2} u(x,t)\right) = (-j\xi)^2 U(\xi,t) = -\xi^2 U(\xi,t)$$

or $\dfrac{\partial}{\partial t} U(\xi,t) = -D\xi^2 U(\xi,t)$

Solution in the time domain: (considering ξ a constant parameter)

$$U(\xi,t) = U(\xi,0)\, e^{-D\xi^2 t}$$

From Fourier back into space x: inverse Fourier transform

$$u(x,t) = \underbrace{\mathcal{F}_x^{-1}}_{\substack{\text{inverse Fourier} \\ \text{from } \xi \text{ to } x}}\left(U(\xi,t)\right) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} U(\xi,t)\, e^{+j\xi x}\, d\xi$$

$$\Rightarrow u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} U(\xi,0)\, e^{-D\xi^2 t}\, e^{+j\xi x}\, d\xi$$

= inverse Fourier of a product:

- $U(\xi,0)$: Fourier of initial conditions $g(x)$:

$$U(\xi,0) = \int_{-\infty}^{+\infty} u(x,0)\, e^{-j\xi x}\, dx$$

$$= \int_{-\infty}^{+\infty} g(x)\, e^{-j\xi x}\, dx$$

- $e^{-D\xi^2 t}$: Fourier of something else, $h(x)$ (impulse response, or "Green's function" — See Week 3)

= convolution of $g(x)$ and $h(x)$!

$$u(x,t) = g(x) * h(x)$$

$$= \int_{-\infty}^{+\infty} g(x_0) \cdot h(x - x_0)\, dx_0$$

RESPONSE @ x, t

INITIAL ACTIVATION @ x_0, $t_0 = 0$

"IMPULSE RESPONSE"; "GREEN'S FUNCTION" effect of activation @ x_0, $t_0=0$ on response @ x, t

$$h(x) = \mathcal{F}_x^{-1}\left(e^{-D\xi^2 t} \right)$$

$$= \frac{1}{2\pi}\underbrace{\int_{-\infty}^{+\infty} e^{-D\xi^2 t} e^{+j\xi x}\, d\xi}$$

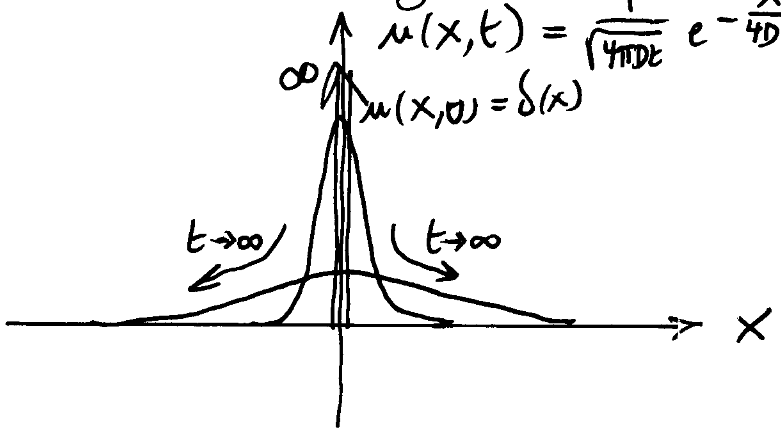
$$e^{-D\xi^2 t + j\xi x} = e^{-Dt\,(\underbrace{\xi^2 - j\frac{x}{Dt}\cdot\xi - \frac{x^2}{4D^2t^2}}_{(\xi - \frac{jx}{2Dt})^2})} \cdot e^{-\frac{x^2}{4Dt}}$$

$$h(x) = \frac{1}{2\pi}\frac{1}{\sqrt{Dt}} e^{-\frac{x^2}{4Dt}} \cdot \underbrace{\int_{-\infty}^{+\infty} e^{-y^2}\, dy}_{\sqrt{\pi}\ (\text{Euler})} = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

$$y = \sqrt{Dt}\left(\xi - \frac{jx}{2Dt}\right)$$

$h(x)$ is also the solution to the diffusion problem with

I.C. $u(x,0) = g(x) = \delta(x)$ impulse @ $x = 0$



Wave equation, on infinite domain, without source term (homogeneous):

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \begin{cases} \text{I.C.}: & u(x,0) = g(x) \\ & \frac{\partial u}{\partial t}(x,0) = \dot{g}(x) \\ \text{B.C}: & u(-\infty,t) = 0 \\ & u(+\infty,t) = 0 \end{cases} \qquad -\infty \leq x \leq +\infty$$

As before, transformed to the Fourier domain in space (x variable):

$$\frac{\partial^2}{\partial t^2} U(\xi,t) = -\underbrace{c^2 \xi^2}_{\geq 0 \,!} U(\xi,t)$$

$\geq 0 \,! \Rightarrow$ conjugate imaginary poles $\pm jc\xi$

Solution in time:

$$U(\xi,t) = \Psi_+(\xi)\, e^{-jc\xi t} + \Psi_-(\xi)\, e^{+jc\xi t}$$

integration "constants";
correspond to forward and
reverse "waves" in the Fourier domain

From Fourier back into space:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Psi_+(\xi)\, \underbrace{e^{-jc\xi t}\, e^{j\xi x}}_{e^{+j\xi(x-ct)}}\, d\xi + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Psi_-(\xi)\, \underbrace{e^{jc\xi t}\, e^{j\xi x}}_{e^{+j\xi(x+ct)}}\, d\xi$$

$$= \underbrace{\psi_+(x-ct)}_{\substack{\text{FORWARD WAVE} \\ \text{velocity } +c}} + \underbrace{\psi_-(x+ct)}_{\substack{\text{REVERSE WAVE} \\ \text{velocity } -c}}$$

where

$\psi_+ = \mathcal{F}_x^{-1}(\Psi_+)$

$\psi_- = \mathcal{F}_x^{-1}(\Psi_-)$

Initial conditions on the forward and reverse waves:

$$u(x,0) = g(x) = \psi_+(x) + \psi_-(x)$$

$$\frac{\partial u}{\partial t}(x,0) = \dot{g}(x) = \frac{1}{c}\left(-\psi_+'(x) + \psi_-'(x)\right)$$

where $\begin{cases} \psi_+'(x) = \frac{d}{dx}\psi_+(x) \\ \psi_-'(x) = \frac{d}{dx}\psi_-(x) \end{cases}$

Let : $h(x) = \int_{-\infty}^{x} \dot{g}(x_0)\,dx_0$

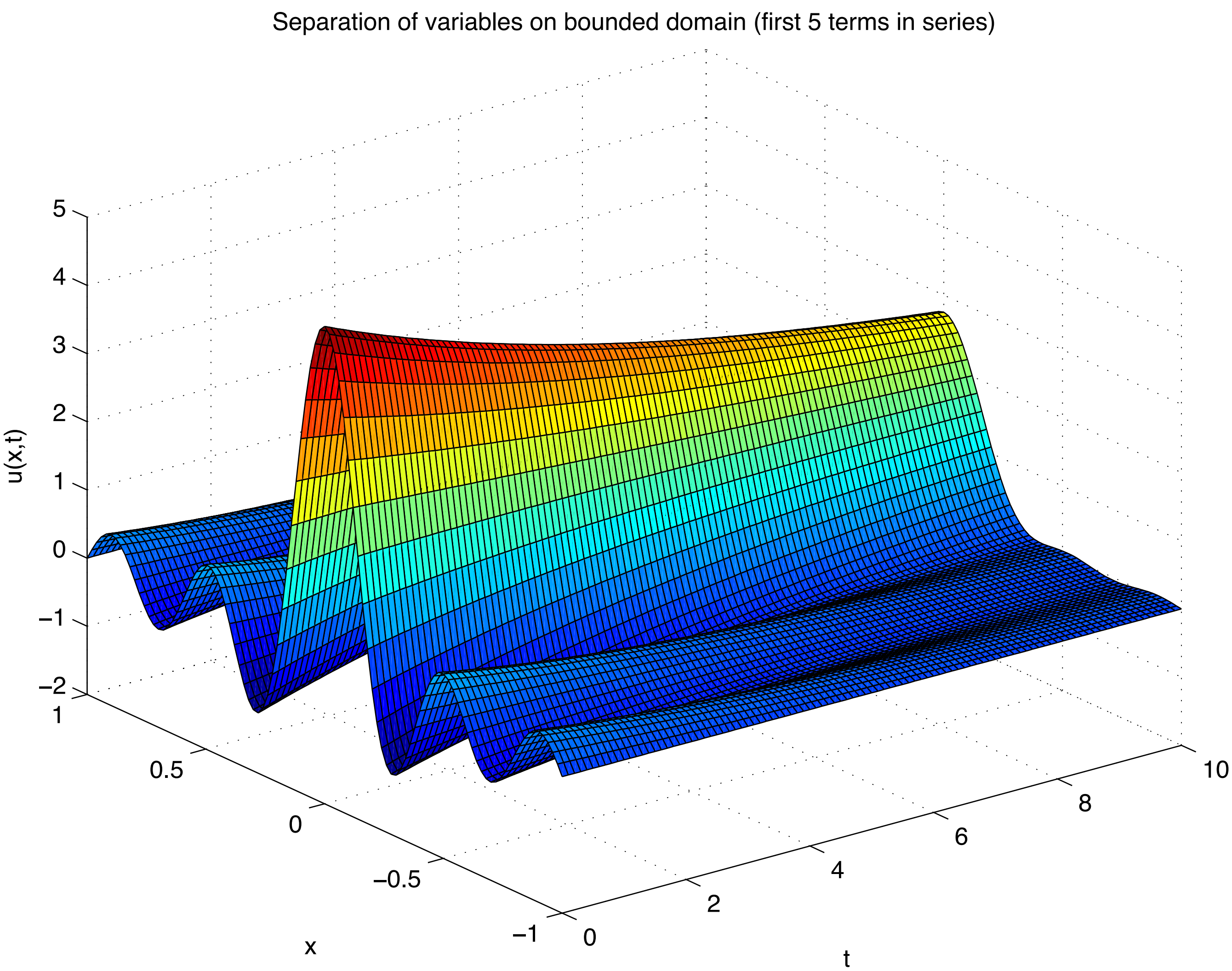
$$\Rightarrow \quad h(x) = \frac{1}{c}\left(-\psi_+(x) + \psi_-(x)\right) + \underset{\substack{\downarrow \\ =0 \text{ because}}}{cst}$$

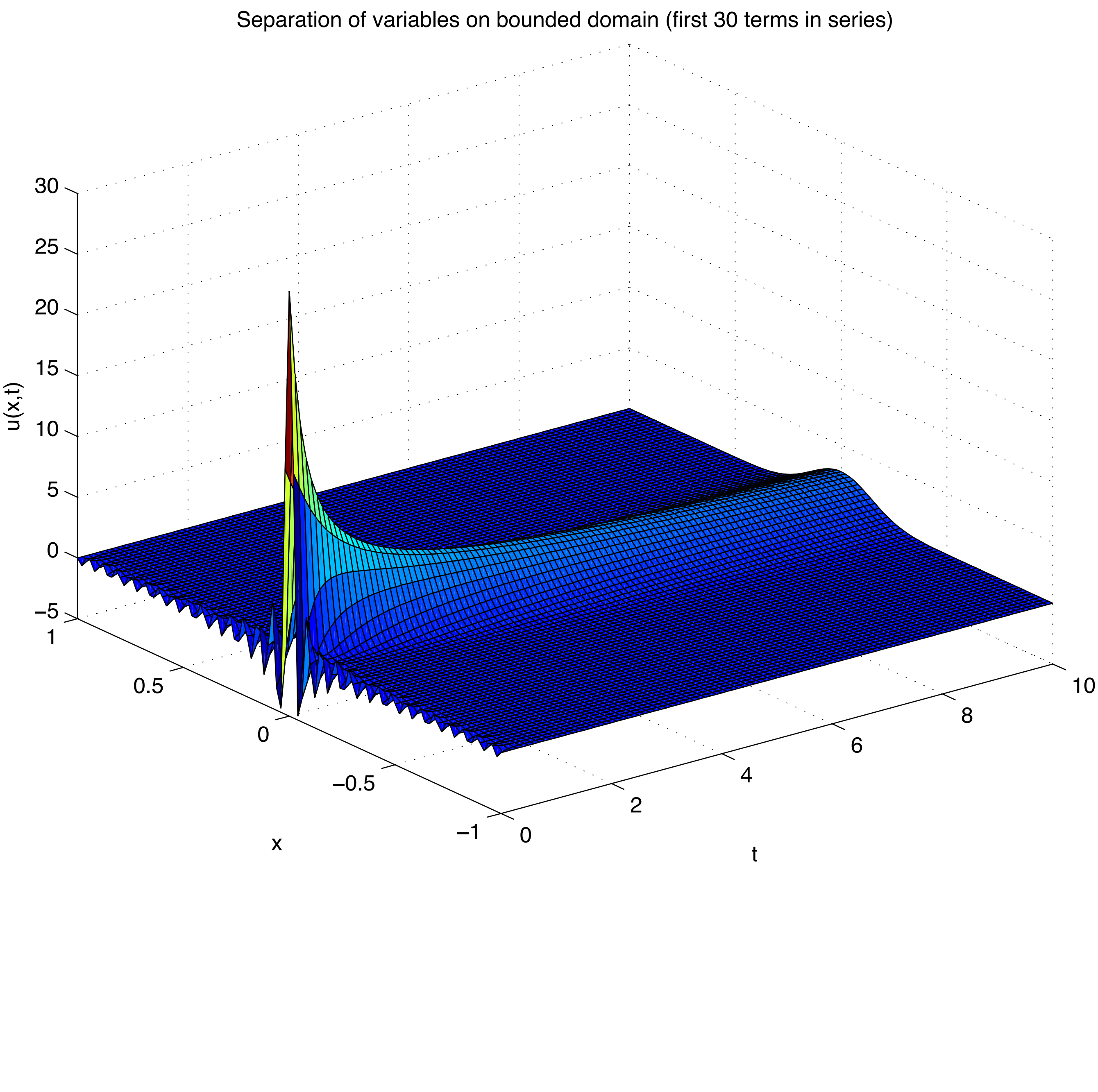
$h(-\infty) = 0$ and
$\psi_+(-\infty) = 0$
$\psi_-(-\infty) = 0$

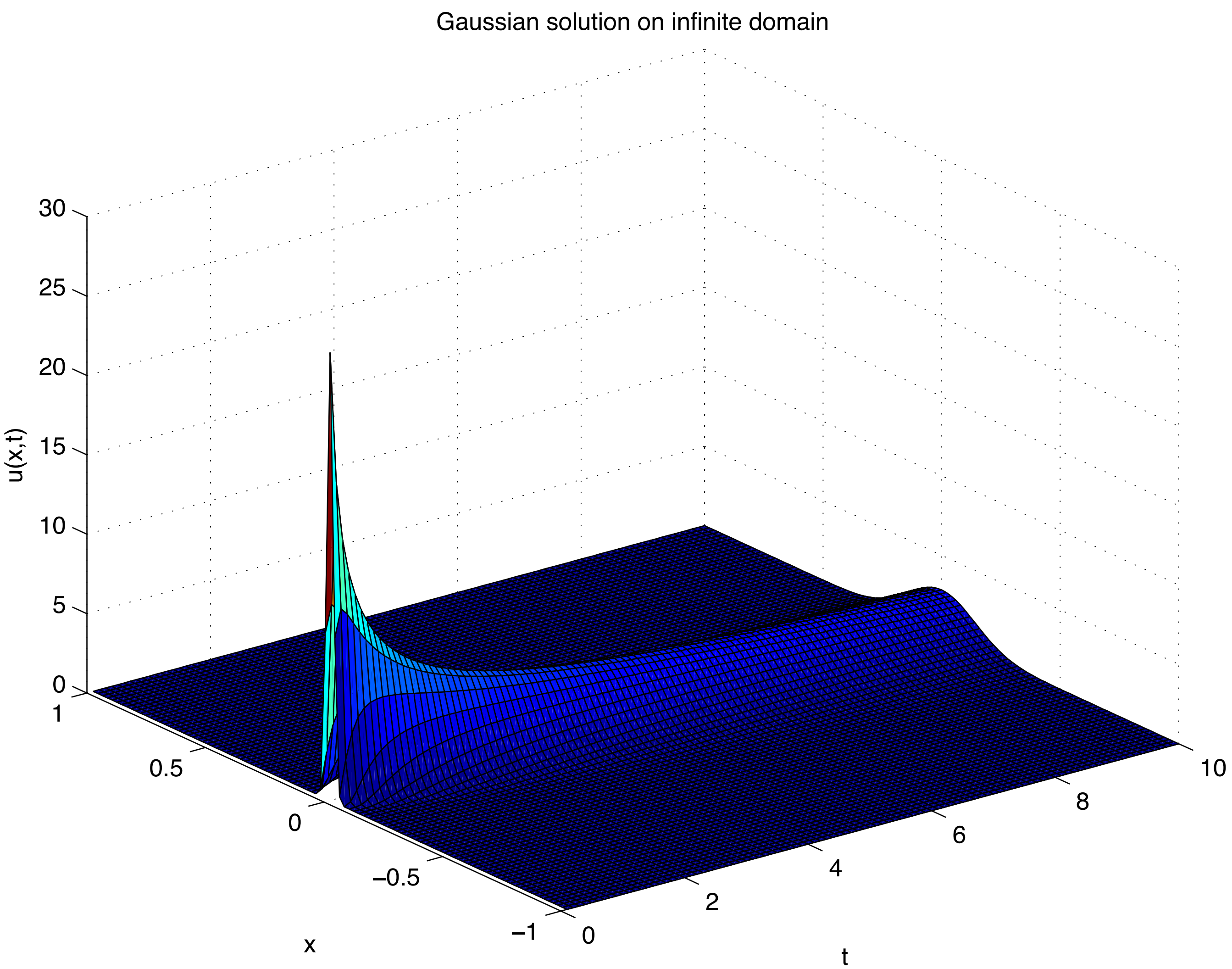
$$\Rightarrow \begin{cases} g(x) = \psi_+(x) + \psi_-(x) \\ c\,h(x) = -\psi_+(x) + \psi_-(x) \end{cases}$$

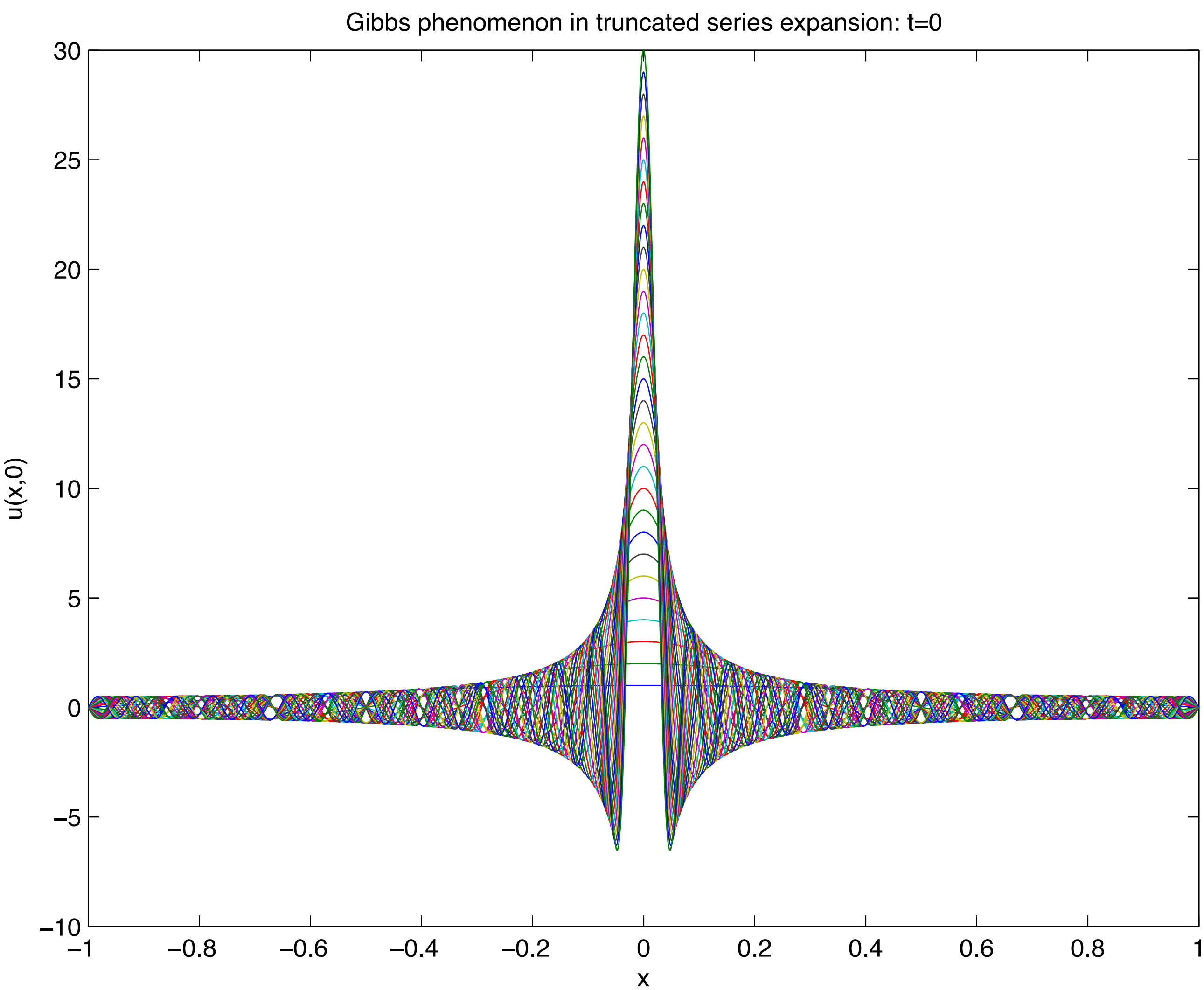
$$\Rightarrow \begin{cases} \psi_+(x) = \frac{1}{2}\left(g(x) - c\,h(x)\right) \\ \psi_-(x) = \frac{1}{2}\left(g(x) + c\,h(x)\right) \end{cases} \quad \text{where } h(x) = \int_{-\infty}^{x} \dot{g}(x_0)\,dx_0$$

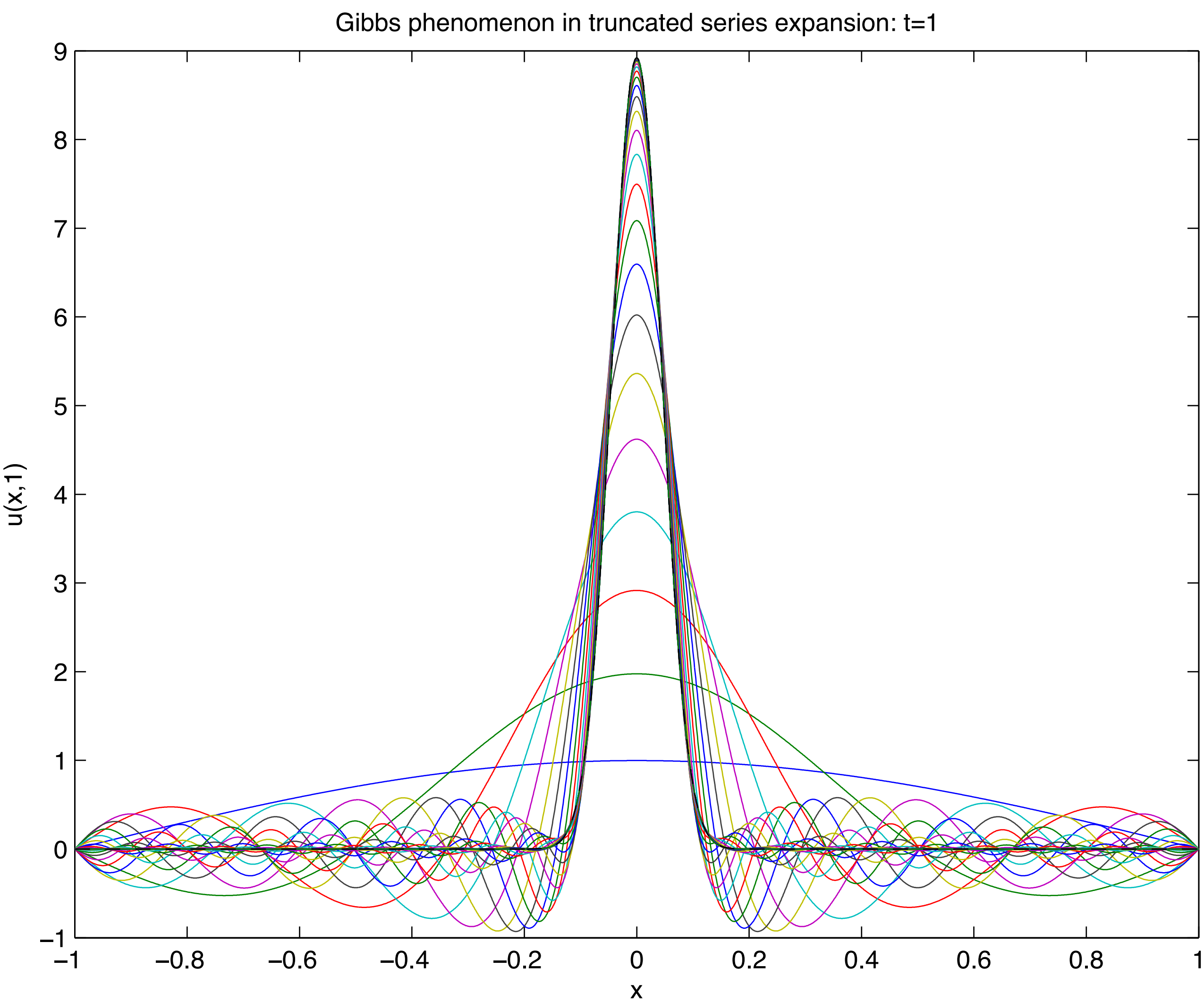
```matlab
function linanal(ns)
% Homogeneous PDE: Linear (1-D) Diffusion
% Analytical solutions on bounded and infinite domain
% BENG 221 example, 10/8/2013
%
% ns: number of terms in the infinite series
%
% e.g.:
% >> linanal(30);
%

% diffusion constant
global D
D = 0.001;

% domain
dx = 0.02; % step size in x dimension
dt = 0.1;  % step size in t dimension
xmesh = -1:dx:1; % domain in x; L/2 = 1
tmesh = 0:dt:10;  % domain in t
nx = length(xmesh); % number of points in x dimension
nt = length(tmesh); % number of points in t dimension


% solution on bounded domain using separation of variables
sol_sep = zeros(nt, nx);
for n = 0:ns-1
    k = (2*n+1)*pi/2; % L = 2
    sol_sep = sol_sep + exp(-D*(k^2)*tmesh)' * cos(k*xmesh);
end

figure(1)
surf(tmesh,xmesh,sol_sep')
title(['Separation of variables on bounded domain (first ', num2str(ns), ' terms in series)'])
xlabel('t')
ylabel('x')
zlabel('u(x,t)')


% solution on infinite domain using Fourier
sol_inf = (4*pi*D*tmesh' * ones(1,nx)).^(-.5) .* exp(-(4*D*tmesh).^(-1)' * xmesh.^2);

figure(2)
surf(tmesh,xmesh,sol_inf')
title('Gaussian solution on infinite domain')
xlabel('t')
ylabel('x')
zlabel('u(x,t)')
```


Separation of variables on bounded domain (first 5 terms in series)
u(x,t)
x
t
5
4
3
2
1
0
−1
−2
1
0.5
0
−0.5
−1
0
2
4
6
8
10

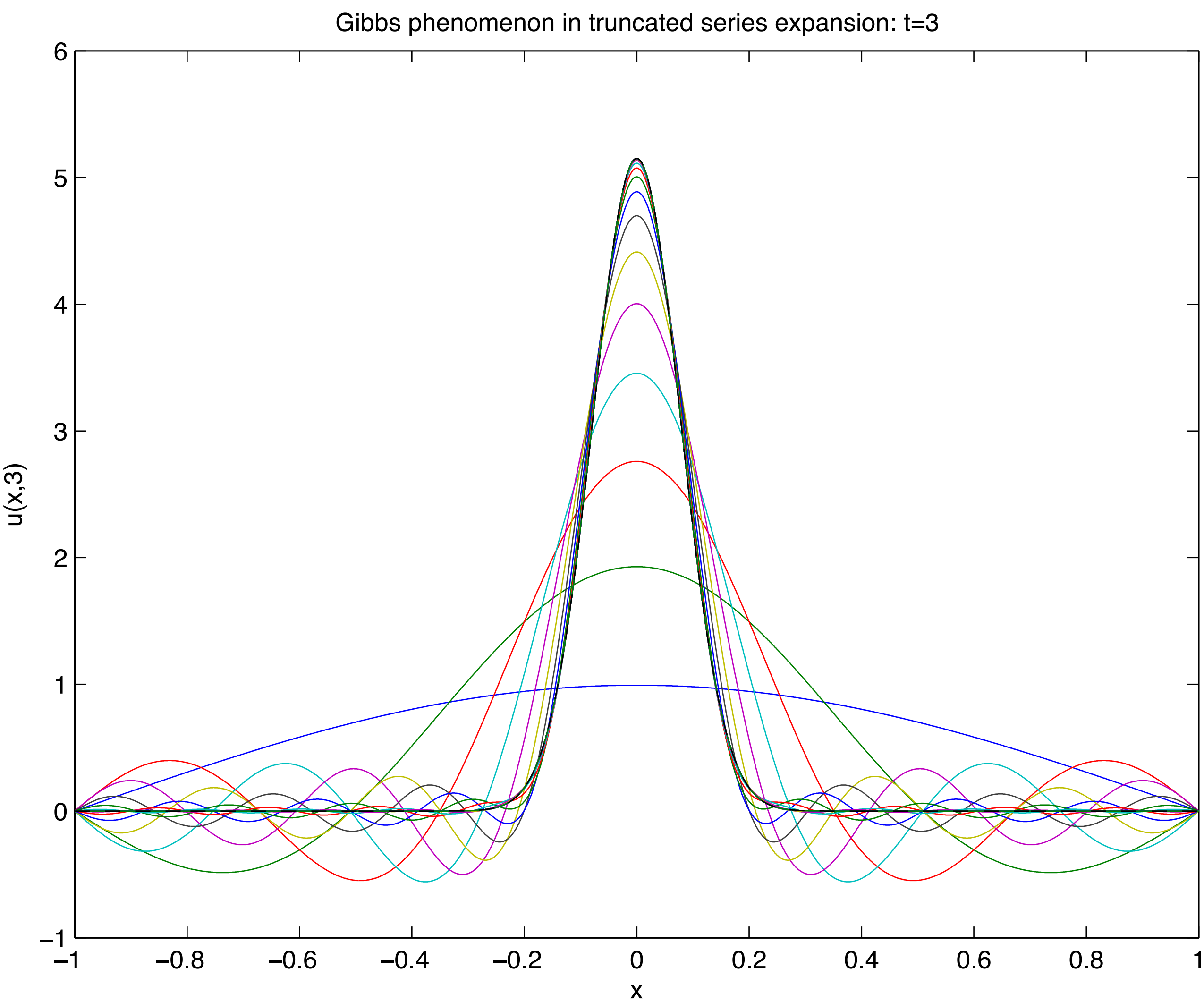

Separation of variables on bounded domain (first 30 terms in series)
u(x,t)
30
25
20
15
10
5
0
−5
1
0.5
0
−0.5
−1
x
0
2
4
6
8
10
t


Gaussian solution on infinite domain
u(x,t)
30
25
20
15
10
5
0
1
0.5
0
−0.5
−1
x
0
2
4
6
8
10
t

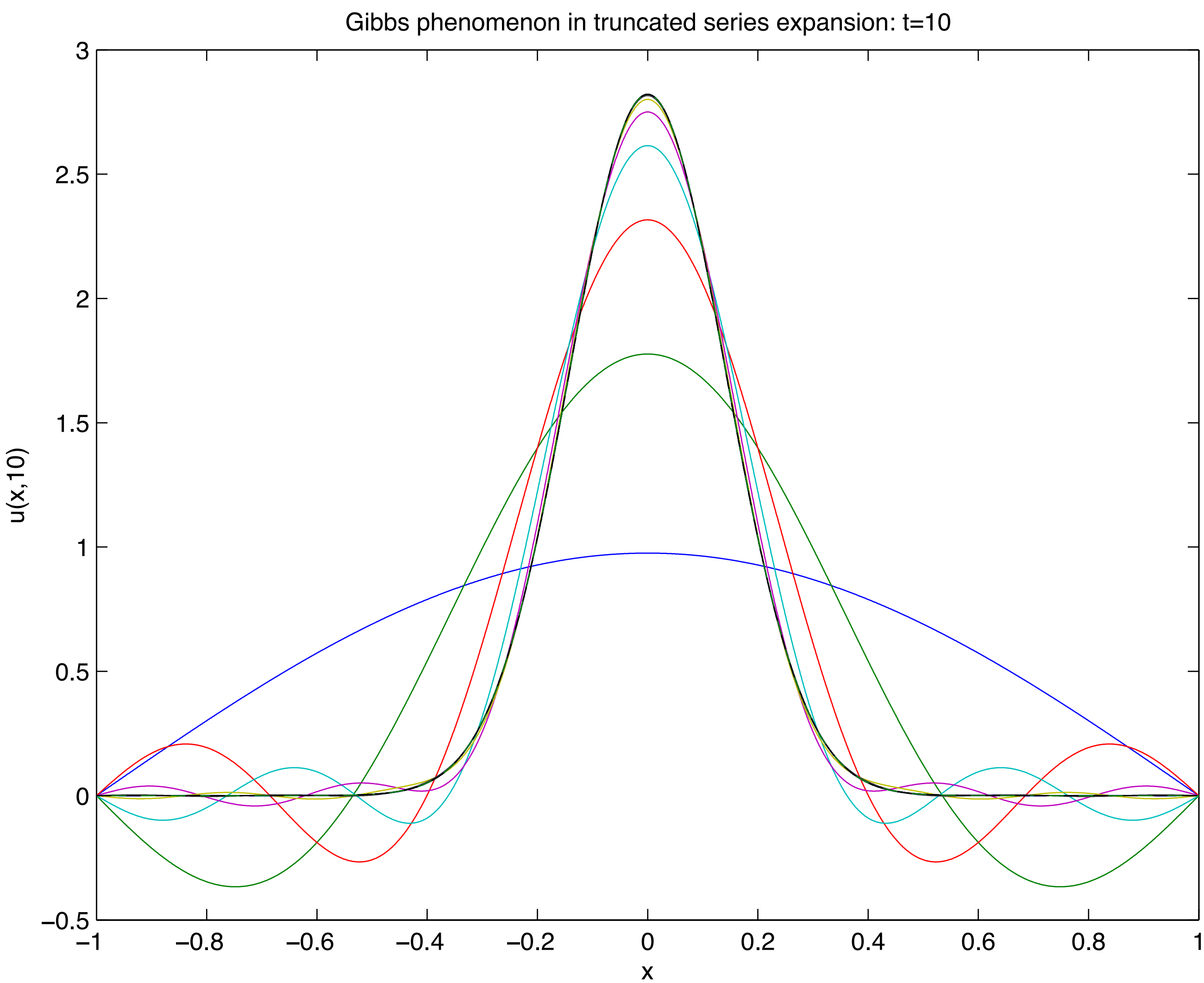

Gibbs phenomenon in truncated series expansion: t=0
u(x,0)
x
30
25
20
15
10
5
0
−5
−10
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1



Gibbs phenomenon in truncated series expansion: t=1
u(x,1)
x
9
8
7
6
5
4
3
2
1
0
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1



Gibbs phenomenon in truncated series expansion: t=3
u(x,3)
x
6
5
4
3
2
1
0
−1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


Gibbs phenomenon in truncated series expansion: t=10
u(x,10)
x
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1