Lecture 18

Wave Transmission of Pressure and Sound

References

Haberman APDE, Ch. 12.

http://en.wikipedia.org/wiki/Wave_equation
http://en.wikipedia.org/wiki/Longitudinal_wave
LONGITUDINAL WAVES IN GASES & LIQUIDS

e.g. (ULTRA) SOUND in 3-D

In COMPRESSIBLE media such as blood, gases and, to some extent, TISSUE, displacement is primarily LONGITUDINAL along the axis of wave propagation:

\[ \vec{u}(x, t) \text{ DISPLACEMENT \ [m]} \]
\[ \vec{p}(x, t) \text{ PRESSURE \ [N/m}^2] \]

Newton:

\[ \text{MASS} \times \text{ACCELERATION} = \text{FORCE} \]

\[ \rho \cdot \frac{\partial \vec{u}}{\partial t} = -\vec{\nabla}p \]

Volume MASS DENSITY \[ \frac{m}{m^3} \]

Density \[ \frac{kg}{m^3} \]

\[ \frac{1}{m} \frac{N}{m^2} \]

FORCE DENSITY = - GRADIENT OF PRESSURE

(pressure = mechanical energy of the compressed medium)
Bulk modulus $B$ (or compressibility $k = \frac{1}{B}$):

$$B = -\frac{1}{\Delta V} \frac{\Delta V}{V}$$

units: $[\frac{N}{m^2}] = [\frac{kg}{m \cdot s^2}]$

pressure change per unit RELATIVE volume change

Volume change $\Delta V$ due to compression of a small volume $V$:

$$\Delta V = \oint_S \tau \cdot n \, dS = \int_\Omega \nabla \cdot \tau \, dV \rightarrow \nabla \cdot \tau \cdot V$$

$$\Rightarrow \frac{\Delta V}{V} = \nabla \cdot \tau$$

$$\Rightarrow \quad p = -B \nabla \cdot \tau = -B \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)$$

$$\Rightarrow \quad \frac{\partial^2 p}{\partial t^2} = -B \nabla \cdot \left( \frac{\partial^2 \tau}{\partial x^2} \right) = -B \nabla \cdot \left( -\nabla \cdot \tau \right)$$

$$= c^2 \nabla \cdot \nabla \cdot \tau = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$

with WAVE VELOCITY $c = \sqrt{\frac{B}{\rho}}$
NOTES:

- All media are, to some extent, COMPRESSIBLE (even solids).

  \[ \rightarrow \text{LONGITUDINAL WAVES with: } C = \sqrt{\frac{B}{\rho}} \rightarrow \text{BULK MODULUS} \]

  \[ \rightarrow \text{TRANSVERSE WAVES with: } C = \sqrt{\frac{G}{\rho}} \rightarrow \text{SHEAR MODULUS} \]

- For a string/cable, propagation and displacement are 1-D
  with: \[ G \rightarrow \frac{T}{A} \]
  \[ \rho \rightarrow \frac{m}{l} \]

- For a perfect gas:
  \[ p \cdot V = RT \Rightarrow \frac{\Delta V}{V} = -\frac{p}{\rho_0} \]
  \[ \Rightarrow B = \rho_0 \]

  \[ \text{change in pressure (volume)} \]
  \[ \text{static pressure (volume)} \]
Transmission of waves in gases. Sound.

Strings present the transmission of transverse waves. In gases waves are transmitted longitudinally. We will analyze the transmission of waves in tube where gas displacements are made by a piston. Moving the piston creates a compression that travels forward. If the piston is quickly retracted then there is a wave of rarefaction that also travels along the tube.

Consider an element of gas in the tube located between $x$ and $x + \Delta x$ where the gas has an equilibrium pressure $p_0$. As the wave advances the element of gas oscillates about its equilibrium position. The coordinate $y$ is used to describe displacements of gas from its equilibrium position. The displacement of the left side of the element of gas has coordinate $y$ and that on the right side $y + \Delta y$. Pressure on the left side is $p$ and on the right side is $p + \Delta p$. For a very thin slice pressure in the displaced gas is $p + p_0$, which is also the pressure on the left side face, and the pressure on the right side face is $p + p_0 + \Delta p$. The forces acting on the element of gas are obtained by multiplying by the area of the tube $A$. The net restoring force acting on the displaced gas is $-\Delta p A$. If $\rho_0$ is the density of the gas at the equilibrium pressure $p_0$ then the mass of element is $\rho_0 A \Delta x$ leading to the equation of motion:

$$\frac{d^2 y}{dt^2} = -\Delta p A$$

Note that $x$ gives the position of the gas molecules at rest (therefore while $p_0$ is uniform in $\Delta x$) while $y$ gives the position of displaced molecules and $p$ is not uniform in $\Delta y$. In the case illustrated since $\Delta y > \Delta x$ we are dealing with a rarefaction wave.
\[ \frac{d^2 y}{dt^2} = -\frac{1}{\rho_0} \Delta p \]

and at the limit for very small \( \Delta x \):

\[ \frac{\partial^2 y}{\partial t^2} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} \quad (23) \]

The volume in its equilibrium position is \( A\Delta x \). In the displaced position the coordinate of the right face is \( x + \Delta x + y + \Delta y \) while the coordinate of the left face is \( x + y \). Therefore the length of the displaced element is given by the difference of these two coordinates or \( \Delta x + \Delta y \) and the change in length, and therefore the change in volume is \( A\Delta y \).

Consider the general definition of compressibility \( k \):

\[ k = -\frac{1}{\text{original volume change in pressure}} \cdot \frac{\text{change in volume}}{\text{change in pressure}} \]

Note that the compressibility of a gas can be derived from the perfect gas equation \( pV = RT \) where

\[ \frac{dV}{dp} = -\frac{RT}{p^2} \quad \text{and} \quad k = -\frac{1}{V \frac{dV}{dp}} = \frac{pRT}{RTp^2} = \frac{1}{p} \]

Referring this definition to our development:

\[ k = -\frac{1}{A\Delta x} \left( \frac{A\Delta y}{(p_0 + p) - p_0} \right) = -\frac{\Delta y}{p\Delta x} \]

therefore:

\[ p = -\frac{1}{k} \frac{\Delta y}{\Delta x} \]

and in the limit:

\[ p = -\frac{1}{k} \frac{\partial y}{\partial x} \quad (24) \]

and in view of (23)
\[
\frac{\partial^2 p}{\partial x^2} = -\frac{1}{k} \frac{\partial^2 y}{\partial x^2}
\]

Therefore substituting in (23) we are led to the one dimensional wave equation for the transmission of longitudinal perturbations:

\[
\frac{\partial^2 y}{\partial t^2} = \frac{1}{k \rho_0} \frac{d^2 y}{dx^2}
\]

The velocity of propagation, by analogy to the wave equation for strings (lateral displacements) is given by:

\[v = \sqrt{\frac{1}{k \rho_0}}\]

The bulk modulus \(B\) is the reciprocal of the compressibility, in other words, the pressure required to induce a volume change relative to the total volume. This quantity is the equivalent to the Young’s modulus \(Y\) for linear changes (stress required to induce a change in strain). Therefore a general expression for the velocity at which waves travel in a materials is:

\[v = \sqrt{\frac{B}{\rho_0}}\]

**Pressure variation in a sound wave**

From the development of the propagation velocity of lateral displacement (waves) in a string we found that a disturbance is propagated with a velocity \(v\), where in this case \(L = \) wave length, and \(A = \) displacement amplitude

\[y = A \cos \left(\frac{2\pi n}{L} (x - vt)\right)\]

If we know the displacement as a function of time \(y(x,t)\) we can compute the pressure by differentiating with respect to \(x\) since:

\[p = -\frac{1}{k} \frac{\partial y}{\partial x}\]

in view of (24) which leads to:
\[ \frac{dy}{dx} = -\frac{2\pi A}{L} \sin \frac{2\pi}{L} (x - vt) \]

and therefore:

\[ p = \frac{2\pi A}{kL} \sin \frac{2\pi}{L} (x - vt) \]

Since \( v = \sqrt{\frac{1}{k\rho_0}} \)

\[ p = \left[ \frac{2\pi\rho_0 v^2 A}{L} \right] \sin \frac{2\pi}{L} (x - vt) \]  \hspace{1cm} (25)

The term within brackets represents the maximal pressure amplitude \( P \) while \( A \) is the maximal displacement.

**Wave velocity & thermodynamics of perfect gases**

The previous equation (25) shows that:

\[ \frac{\partial p}{\partial \rho} = \frac{2\pi A}{L} v^2 \sin \frac{2\pi}{L} (x - vt) \]

\hspace{1cm} (26)

therefore using the thermodynamic relation:

\[ pV = nRT = n \frac{M}{M} RT \quad \text{where} \quad \frac{nM}{V} = \rho \quad \text{then} \quad p = \frac{\rho RT}{M} \]

therefore in terms of maximum values (26) is also equal to:

\[ \frac{\partial p}{\partial \rho} = \frac{RT}{M} = v^2 \]

a result derived by Newton, which underestimates the actual speed by about 15%. The more correct formulation was given by Laplace, who realized that the compression and relaxation in the sound wave is too rapid for allowing constant temperature (isothermal conditions), and that the actual conditions were adiabatic, i.e., no heat transfer due to the high speed at which compression and rarefaction occur, leading to the expression:

\[ v = \sqrt{\gamma \frac{RT}{M}} \]
where $\gamma = \frac{C_p}{C_v}$

which is the ratio of specific heat at constant pressure vs. constant volume, usually about 1.4.

**Pressure dispersion**

The analytical derivation does not include a mechanism for the attenuation of the pressure amplitude, which occurs due to refraction and absorption of the pressure wave. This can be accounted for by the expression:

$$A = A_0 e^{-\alpha x}$$

where $\alpha$ is a parameter that characterizes the viscous effects in the medium, or the conversion of mechanical energy in the wave into thermal energy.

Pressure waves that originate from a point source decay naturally at the rate of $-60 \log R \text{ db}$, where $R$ is the ratio of radial distances, and for cylindrical sources at the rate of $-40 \log R$.

The dispersion of pressure can also be described by the diffusion equation:

$$\frac{\partial p}{\partial t} = K \nabla^2 p$$

**Intensity of sound waves**

Waves propagate energy. The intensity $I$ of a traveling wave is defined as the average rate energy is transported by the wave per unit area across a surface perpendicular to the direction of propagation. Also intensity is the average power transported per unit area. The energy associated with a travelling wave is in part potential, associated with the compression of the medium, and kinetic relate to particle velocity. By analogy to the dynamics and energy distribution of the spring mass system, the total energy is constant in time (no dissipation) ad we can calculate intensity by considering only pressure effects.

Work done in the compression process is:

$$W = -\int p \, dv$$

and introducing the definition for compressibility:
\[ dv = -k v_0 dp \hat{\alpha} \quad \text{therefore} \quad W = -k v_0 \int p dp \]

Integrating between 0 and the maximum pressure change \( P \) defines the pressure energy per unit volume, which is the same as the total energy per unit volume:

\[
\frac{W}{v_0} = \frac{1}{2} k P^2
\]

The sound energy in the volume traveling with a wave, or energy crossing per unit area per unity time equals the energy in the volume element \( Av \Delta t \) divided by \( A \):

\[
I = \frac{1}{2} k P^2 v = \frac{P^2}{2 \rho v}
\]