Lecture 10

General Solutions to Inhomogeneous PDEs using Green's Functions

References

See also Lecture 7 notes.

Haberman APDE, Sec. 11.3.

http://en.wikipedia.org/wiki/Green's_function

3) INHOMOGENEOUS PDEs WITH INHOMOGENEOUS B.C.

e.g. $$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \underset{\substack{\uparrow \\ \text{DRIVING} \\ \text{SOURCE}}}{Q(x,t)} \quad \text{with} \quad \begin{cases} u(0,t) = u_0(t) & \text{B.C (VALUE)} \\ u(L,t) = u_L(t) & \text{B.C (VALUE)} \\ u(x,0) = g(x) & \text{I.C.} \end{cases}$$

Let $$u(x,t) = \underset{\substack{\downarrow \\ \text{HOMOGENEOUS B.C.}}}{u_H(x,t)} + \underset{\substack{\downarrow \\ \text{INHOMOGENEOUS B.C.}}}{u_I(x,t)} \qquad (1)$$

For now, assume any $u_I(x,t)$ that satisfies the B.C., as the Green's solution does not depend on it.

$$\begin{cases} u_I(0,t) = u_0(t) & \text{B.C.} \\ u_I(L,t) = u_L(t) & \text{B.C.} \end{cases} \quad \text{and} \quad u_I(x,0) = 0 \quad \text{I.C.}$$

Homogeneous B.C. PDE for $u_H(x,t)$:

$$\frac{\partial u_H}{\partial t} = k\frac{\partial^2 u_H}{\partial x^2} + Q_H(x,t) \quad \text{with} \quad \begin{cases} u_H(0,t) = 0 & \text{(VALUE)} \\ u_H(L,t) = 0 & \text{(VALUE)} \\ u_H(x,0) = g(x) \end{cases}$$

Substitution of (1) in the inhomogeneous PDE gives:

$$Q(x,t) = Q_H(x,t) + \frac{\partial}{\partial t}u_I(x,t) - k\frac{\partial^2}{\partial x^2}u_I(x,t)$$

or $$Q_H = Q - \left(\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}\right)u_I$$

Substitution of the homogeneous solution using Green's functions:

$$u(x,t) = u_H(x,t) + u_I(x,t)$$

$$= \int_0^L g(x_0)G(x,t;x_0,0)\,dx_0 + \int_0^L\int_0^t Q_H(x_0,t_0)G(x,t;x_0,t_0)\,dx_0\,dt_0 + u_I(x,t)$$

$$= \int_0^L g(x_0)G(x,t;x_0,0)\,dx_0 + \int_0^L\int_0^t Q(x_0,t_0)G(x,t;x_0,t_0)\,dx_0\,dt_0$$

$$(*)\left\{ \quad - \int_0^L\int_0^t \Big(\underbrace{\frac{\partial}{\partial t_0}}_{T} - k\underbrace{\frac{\partial^2}{\partial x_0^2}}_{X}\Big)u_I(x_0,t_0)\cdot G(x,t;x_0,t_0)\,dx_0\,dt_0 + \underbrace{\cancel{u_I(x,t)}}_{(\dagger)}\right.$$

Simplify $(*)$:

T: $-\int_0^L f_T\,dx_0$ with $f_T(x_0) = \int_0^t \frac{\partial}{\partial t_0} u_I(x_0,t_0)G(x,t;x_0,t_0)\,dt_0$

Integration by parts (in t_0):

$$f_T = \Big[u_I(x_0,t_0)\,G(\cdot,\cdot;\cdot,t_0)\Big]_0^t - \int_0^t u_I(x_0,t_0)\cdot\frac{\partial}{\partial t_0}G(\cdot,\cdot;\cdot,t_0)\,dt_0$$

$$\Rightarrow T = \underbrace{-\int_0^L u_I(x_0,t)\underbrace{G(x,t;x_0,t)}_{=\,\delta(x-x_0)}\,dx_0}_{=\,\cancel{u_I(x,t)}\ (\dagger)} + \cancel{\int_0^L u_I(x_0,0)\cdot \underbrace{G(x,t;x_0,0)}_{=\,0}\,dx_0} \quad \text{(causality)}$$

$$+ \int_0^L\int_0^t u_I(x_0,t_0)\frac{\partial}{\partial t_0}G(x,t;x_0,t_0)\,dx_0\,dt_0$$

X: $k\int_0^t f_X\,dt_0$ with $f_X(t_0) = \int_0^L \frac{\partial^2}{\partial x_0^2}u_I(x_0,t_0)G(x,t;x_0,t_0)\,dx_0$

Integration by parts, twice in x_0:

$$\int X = \int_0^L \frac{\partial^2}{\partial x_0^2} \mu_I(x_0,t_0)\, G(x,t;x_0,t_0)\, dx_0$$

$$= \Big[\frac{\partial}{\partial x_0}\mu_I(x_0,\cdot)\, \cancel{G(\cdot,\cdot;x_0,\cdot)}\Big]_0^L - \int_0^L \frac{\partial}{\partial x_0}\mu_I(x_0,\cdot)\frac{\partial}{\partial x_0} G(\cdot,\cdot;x_0,\cdot)\,dx_0$$

$\| \; 0$ because of B.C.

$$= -\Big[\mu_I(x_0,\cdot)\frac{\partial}{\partial x_0} G(\cdot,\cdot;x_0,\cdot)\Big]_0^L + \int_0^L \mu_I(x_0,\cdot)\frac{\partial^2}{\partial x_0^2} G(\cdot,\cdot;x_0,\cdot)\,dx_0$$

$$\Rightarrow (*) = X + T + \mu_I(x,t)$$

$$= -k\int_0^t \Big[\mu_I(x_0,t_0)\frac{\partial}{\partial x_0} G(x,t;x_0,t_0)\Big]_0^L dt_0$$

$$+ \int_0^L\int_0^t \mu_I(x_0,t_0)\cdot \cancel{\underbrace{\left(\frac{\partial}{\partial t_0} + k\frac{\partial^2}{\partial x_0^2}\right) G(x,t;x_0,t_0)}_{=0 \text{ because of RECIPROCITY}}}\, dx_0\, dt_0$$

$$= \int_0^t \underbrace{\mu_0(t_0)}_{\downarrow\ \text{B.C. @ } x=0}\, k\frac{\partial}{\partial x_0} G(x,t;0,t_0)\,dt_0 - \int_0^t \underbrace{\mu_L(t_0)}_{\downarrow\ \text{B.C @ } x=L}\, k\frac{\partial}{\partial x_0} G(x,t;L,t_0)\,dt_0$$

independent of chosen form of $\mu_I(x_0,t_0)$

$$\Rightarrow u(x,t) = \int_0^L \underbrace{g(x_0)}_{\text{I.C.}} \underbrace{G(x,t;x_0,0)}_{\text{Green's @ } t_0=0} dx_0 + \int_0^L\!\!\int_0^t \underbrace{Q(x_0,t_0)}_{\substack{\text{DRIVING}\\ \text{SOURCE}}} \underbrace{G(x,t;x_0,t_0)}_{\text{Green's}} dx_0\, dt_0$$

$$+ \int_0^t \underbrace{u_0(t_0)}_{\text{B.C. @ 0}} \underbrace{k \frac{\partial}{\partial x_0} G(x,t;0,t_0)}_{\substack{\text{Green's INFLUX}\\ \text{@ } x_0=0}} dt_0 - \int_0^t \underbrace{u_L(t_0)}_{\text{B.C. @ } L} \underbrace{k \frac{\partial}{\partial x_0} G(x,t;L,t_0)}_{\substack{\text{Green's OUTFLUX}\\ \text{@ } x_0=L}} dt_0$$

where $G(x,t;x_0,t_0)$ is the same Green's function as found for the HOMOGENEOUS PDE

Note: the form of this solution changes based on the type of B.C.:

e.g. FLUX based B.C. : $\begin{cases} \dfrac{du}{dx}(0,t) = u_0'(t) \\ \dfrac{du}{dx}(L,t) = u_L'(t) \end{cases}$

$$\Rightarrow u(x,t) = \quad \ldots$$

$$- \int_0^t \underbrace{k\, u_0'(t_0)}_{\substack{\text{OUTFLUX B.C.}\\ \text{@ 0}}} \underbrace{G(x,t;0,t_0)}_{\substack{\text{Green's}\\ \text{@ } x_0=0}} dt_0 + \int_0^t \underbrace{k\, u_L'(t_0)}_{\substack{\text{INFLUX B.C.}\\ \text{@ } L}} \underbrace{G(x,t;L,t_0)}_{\substack{\text{Green's}\\ \text{@ } x_0=L}} dt_0$$

GREEN'S FUNCTION FOR PDEs ON UNBOUNDED DOMAINS

e.g.: $$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \underbrace{Q(x,t)}_{\text{DRIVING SOURCE}} \quad \text{with} \begin{cases} u(-\infty,t)=0 & \text{B.C.} \\ u(+\infty,t)=0 & \text{B.C.} \\ u(x,0)=g(x) & \text{I.C.} \end{cases}$$

$$\Rightarrow u(x,t) = \underbrace{\int_{-\infty}^{+\infty} g(x_0)\,G(x,t;x_0,0)\,dx_0}_{\text{I.C.}} + \underbrace{\int_{-\infty}^{+\infty}\int_0^t Q(x_0,t_0)\,G(x,t;x_0,t_0)\,dx_0\,dt_0}_{\text{DRIVING SOURCE}}$$

with $$\frac{\partial G(x,t;x_0,t_0)}{\partial t} = k\frac{\partial^2 G(x,t;x_0,t_0)}{\partial x^2} + \delta(x-x_0)\,\delta(t-t_0)$$

Solution using Laplace in time, and Fourier in space:

$$G(x,t;x_0,t_0) = \frac{1}{2\sqrt{\pi k(t-t_0)}} \cdot e^{-\frac{(x-x_0)^2}{4k(t-t_0)}}$$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad u(x,0) = \delta(x) \quad \text{and} \quad \begin{cases} u(-\infty,t) = 0 \\ u(+\infty,t) = 0 \end{cases}$$

(INFINITE DOMAIN)

- Fourier in x: $u(x,t) \longrightarrow u_{\mathcal{F}}(\omega,t)$

$$\frac{\partial u_{\mathcal{F}}}{\partial t} = -k\omega^2 u_{\mathcal{F}} \quad \text{with} \quad u_{\mathcal{F}}(\omega,0) = 1$$

- Laplace in t: $u_{\mathcal{F}}(\omega,t) \rightarrow u_{\mathcal{FL}}(\omega,s)$

$$s\,u_{\mathcal{FL}} - \underbrace{u_{\mathcal{F}}(\omega,0)}_{=1} = -k\omega^2 u_{\mathcal{FL}}$$

or $$u_{\mathcal{FL}} = \frac{1}{s+k\omega^2}$$

- Inverse Laplace in s: $u_{\mathcal{FL}}(\omega,s) \longrightarrow u_{\mathcal{F}}(\omega,t)$

$$u_{\mathcal{F}} = e^{-k\omega^2 t}$$

- Inverse Fourier in ω: $u_{\mathcal{F}}(\omega,t) \rightarrow u(x,t)$

$\left(\sigma\sqrt{2\pi}\, e^{-\sigma^2\omega^2/2} \rightarrow e^{-x^2/2\sigma^2} \right)$

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}}\, e^{-\frac{x^2}{4kt}} = G(x,t;0,0) \qquad (t \geq 0)$$

$$\Rightarrow G(x,t;x_0,t_0) = \frac{1}{\sqrt{4\pi k(t-t_0)}}\, e^{-\frac{(x-x_0)^2}{4k(t-t_0)}} \qquad (t \geq t_0)$$

Generalized to 3 dimensions (using Green's Theorem):

- HEAT/DIFFUSION EQ. (pp. 523–529) : $\dfrac{\partial u}{\partial t} = k\,\vec{\nabla}_{\vec{x}}^{2} u + Q(\vec{x},t)$

$$u(\vec{x},t) = \int_0^t \iiint G(\vec{x},t;\vec{x}_0,t_0)\,\underset{\substack{\downarrow\\ \text{DRIVING SOURCE}}}{Q(\vec{x}_0,t_0)}\,d^3\vec{x}_0\,dt_0 + \iiint G(\vec{x},t;\vec{x}_0,0)\,\underset{\substack{\downarrow\\ \text{I.C.}}}{u(\vec{x}_0,0)}\,d^3\vec{x}_0$$

$$+\, k\int_0^t \oint\!\!\oint \Big(\underset{\substack{\| \\ 0 \\ \text{for VALUE B.C.} \\ \text{or } \infty \text{ domain}}}{G(\vec{x},t;\vec{x}_0,t_0)}\,\vec{\nabla}_{\vec{x}_0}\underset{\substack{\downarrow\\ \text{FLUX B.C.}}}{u(\vec{x}_0,t_0)} - \underset{\substack{\downarrow\\ \text{VALUE B.C.}}}{u(\vec{x}_0,t_0)}\,\underset{\substack{\| \\ 0 \\ \text{for FLUX B.C.} \\ \text{or } \infty \text{ domain}}}{\vec{\nabla}_{\vec{x}_0} G(\vec{x},t;\vec{x}_0,t_0)}\Big)\cdot\vec{n}\,dS_0\,dt_0$$

Infinite domain (p. 531) : $G(\vec{x},t;\vec{x}_0,t_0) = \dfrac{1}{(4\pi k(t-t_0))^{3/2}}\, e^{-\frac{\|\vec{x}-\vec{x}_0\|^2}{4k(t-t_0)}}$

- WAVE EQ. (pp. 508–515) : $\dfrac{\partial^2 u}{\partial t^2} = c^2\,\vec{\nabla}_{\vec{x}}^{2} u + Q(\vec{x},t)$

$$u(\vec{x},t) = \int_0^t \iiint G(\vec{x},t;\vec{x}_0,t_0)\,\underset{\substack{\downarrow\\ \text{DRIVING SOURCE}}}{Q(\vec{x},t_0)}\,d^3\vec{x}_0\,dt_0$$

$$+\iiint \Big[\underset{\substack{\downarrow\\ \text{CHANGE I.C.}}}{\frac{\partial u}{\partial t_0}(\vec{x}_0,0)}\, G(\vec{x},t;\vec{x}_0,0) - \underset{\substack{\downarrow\\ \text{VALUE I.C.}}}{u(\vec{x}_0,0)}\,\frac{\partial}{\partial t_0} G(\vec{x},t;\vec{x}_0,0)\Big]\, d^3\vec{x}_0$$

$$-\,c^2\int_0^t \oint\!\!\oint \Big[\underset{\substack{\downarrow\\ \text{VALUE B.C.}}}{u(\vec{x}_0,t_0)}\,\vec{\nabla}_{\vec{x}_0} G(\vec{x},t;\vec{x}_0,t_0) - G(\vec{x},t;\vec{x}_0,t_0)\,\vec{\nabla}_{\vec{x}_0}\underset{\substack{\downarrow\\ \text{FLUX B.C.}}}{u(\vec{x}_0,t_0)}\Big]\cdot\vec{n}\,dS_0\,dt_0$$

Infinite domain (pp. 518–519) : $G(\vec{x},t;\vec{x}_0,t_0) = \dfrac{\delta(\rho - c(t-t_0))}{4\pi c\rho}$ with $\rho = \|\vec{x}-\vec{x}_0\|$