BENG 221
Mathematical Methods in Bioengineering

Lecture 1

Introduction

ODEs and Linear Systems

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Course Objectives

1. Acquire methods for quantitative analysis and prediction of biophysical processes involving spatial and temporal dynamics:
   - Derive partial differential equations from physical principles;
   - Formulate boundary conditions from physical and operational constraints;
   - Use engineering mathematical tools of linear systems analysis to find a solution or a class of solutions;
2. Learn to apply these methods to solve engineering problems in medicine and biology:
   - Formulate a bioengineering problem in quantitative terms;
   - Simplify (linearize) the problem where warranted;
   - Solve the problem, interpret the results, and draw conclusions to guide further design.
3. Enjoy!

Today's Coverage:

Ordinary Differential Equations

Linear Time-Invariant Systems

Eigenmodes

Convolution and Response Functions

ODE Problem Formulation

Solve for the dynamics of n variables $x_1(t), x_2(t), \ldots x_n(t)$ in time (or other ordinate) t described by m differential equations:

ODE

$$\begin{aligned} \mathcal{F}_i \Bigg(x_1, \frac{dx_1}{dt}, \ldots \frac{d^k x_1}{dt^k}, \ldots \\ x_2, \frac{dx_2}{dt}, \ldots \frac{d^k x_2}{dt^k}, \ldots \\ x_n, \frac{dx_n}{dt}, \ldots \frac{d^k x_n}{dt^k} \Bigg) &= 0 \end{aligned} \tag{1}$$

for $i = 1, \ldots m$, where $m \leq n$ and $k \leq n$. Solutions are generally not unique. A unique solution, or a reduced set of solutions, is determined by specifying initial or boundary conditions on the variables.

ODE Examples

Kinetics of mass m with potential $V(x)$:

$$\frac{1}{2} m \left(\frac{dx}{dt} \right)^2 + V(x) = 0 \tag{2}$$

Two masses with coupled potential $V(x)$:

$$\frac{1}{2} m_1 \left(\frac{dx_1}{dt} \right)^2 + \frac{1}{2} m_2 \left(\frac{dx_2}{dt} \right)^2 + V(x_1, x_2) = 0 \tag{3}$$

Second order nonlinear ODE:

$$x \frac{d^2 x}{dt^2} = \frac{1}{2} \left(\frac{dx}{dt} \right)^2 \tag{4}$$

ODE in Canonical Form

In *canonical form*, a set of *n* ODEs specify the first order derivatives of each of *n* single variables in the other variables, without coupling between derivatives or to higher order derivatives:

Canonical ODE

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1, x_2, \ldots x_n) \\ \frac{dx_2}{dt} &= f_2(x_1, x_2, \ldots x_n) \\ &\vdots \\ \frac{dx_n}{dt} &= f_n(x_1, x_2, \ldots x_n). \end{aligned} \tag{5}$$

Not every system of ODEs can be formulated in canonical form. An important class of ODEs that can be formulated in canonical form are *linear ODEs*.

Canonical ODE Examples

Amplitude stabilized quadrature oscillator:

$$\begin{cases} \frac{dx}{dt} = -y - (x^2 + y^2 - 1)\ x \\ \frac{dy}{dt} = x - (x^2 + y^2 - 1)\ y \end{cases} \tag{6}$$

Any first-order canonical ODE without explicit time dependence can be solved by separation of variables, *e.g.*,

$$\frac{dx}{dt} = (1 + x^2)/x \tag{7}$$

Initial and Boundary Conditions

Initial conditions are values for the variables, and some of their derivatives of various order, specified at one initial point in time t_0, *e.g.*, $t = 0$:

IC

$$\frac{d^i x_j}{dt^i}(0) = c_{ij}, \qquad i = 0, \ldots m, \quad j = 1, \ldots n. \tag{8}$$

Boundary conditions are more general conditions linking the variables, and/or their first and higher derivatives, at one or several points in time t_k:

BC

$$g_l(\ldots, \frac{d^i x_j}{dt^i}(t_k), \ldots) = 0. \tag{9}$$

ICs in Canonical Form

For ODEs in canonical form, initial conditions for each of the variables are specified at initial time t_0, *e.g.*, $t = 0$:

Canonical IC

$$\begin{aligned} x_1(0) &= c_1 \\ x_2(0) &= c_2 \\ &\vdots \\ x_n(0) &= c_n \end{aligned} \tag{10}$$

ICs for first or higher order derivatives are not required for canonical ODEs.

Linear Canonical ODEs

Linear time-invariant (LTI) systems can be described by linear canonical ODEs with constant coefficients:

LTI ODE

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\,\mathbf{x} + \mathbf{b} \tag{11}$$

with $\mathbf{x} = (x_1, \ldots x_n)^T$, and with linear initial conditions:

LTI IC

$$\mathbf{x}(0) = \mathbf{e} \tag{12}$$

or linear boundary conditions at two, or more generally several, time points:

LTI BC

$$\mathbf{C}\,\mathbf{x}(0) + \mathbf{D}\,\mathbf{x}(T) = \mathbf{e} \tag{13}$$

LTI Systems ODE Examples

Examples abound in biomechanical and electromechanical systems (including cardiovascular system, and MEMS biosensors), and more recently bioinformatics and systems biology.
A classic example is the *harmonic oscillator* ($k = 0$), and more generally the *damped oscillator* or *resonator*:

$$\left\{ \begin{array}{rcl} \frac{du}{dt} & = & v \\ m\frac{dv}{dt} & = & -k\, u - \gamma\, v + f_{ext} \end{array} \right. \tag{14}$$

where u represents some physical form of deflection, and v its velocity. Typical parameters include mass/inertia m, stiffness k, and friction γ. The *inhomogeneous* term f_{ext} represents an external force acting on the resonator.

LTI Homogeneous ODEs

In general, LTI ODEs are *inhomogeneous*. *Homogeneous* LTI ODEs are those for which $\mathbf{x} \equiv 0$ is a valid solution. This is the case for LTI ODEs with zero driving force $\mathbf{b} = 0$ and zero IC/BC:

LTI Homogeneous ODE

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\,\mathbf{x} \tag{15}$$

LTI Homogeneous IC

$$\mathbf{C}\,\mathbf{x}(0) = 0 \tag{16}$$

LTI Homogeneous BC

$$\mathbf{C}\,\mathbf{x}(0) + \mathbf{D}\,\mathbf{x}(T) = 0. \tag{17}$$

Eigenmodes, arbitrarily scaled non-trivial solutions $\mathbf{x} \neq 0$, exist for under-determined IC/BC (rank-deficient $\mathbf{C}$ and $\mathbf{D}$).

Eigenmode Analysis

Eigenvalue-eigenvector decomposition of the matrix **A** yields the eigenmodes of LTI homogeneous ODEs. Let:

$$\mathbf{A}\,\mathbf{x}_i = \lambda_i\,\mathbf{x}_i \tag{18}$$

with eigenvectors $\mathbf{x}_i$ and corresponding eigenvalues λ_i. Then

Eigenmodes

$$\mathbf{x}(t) = c_i\,\mathbf{x}_i\,e^{\lambda_i t} \tag{19}$$

are *eigenmode* solutions to the LTI homogeneous ODEs (15) for any scalars c_i. There are n such eigenmodes, where n is the rank of **A** (typically, the number of LTI homogeneous ODEs).

Orthonormality and Inhomogeneous IC/BCs

The general solution is expressed as a linear combination of eigenmodes:

$$\mathbf{x}(t) = \sum_{i=1}^{n} c_i \, \mathbf{x}_i \, e^{\lambda_i t} \tag{20}$$

For *symmetric* matrix A ($A_{ij} = A_{ji}$) the set of eigenvectors $\mathbf{x}_i$ is *orthonormal*:

$$\mathbf{x}_i^T \mathbf{x}_j = \delta_{ij} \tag{21}$$

so that the solution to the homogeneous ODEs (15) with inhomogeneous ICs (12) reduces to $c_i = \mathbf{x}_i^T \, \mathbf{x}(0)$, or:

LTI inhomogenous IC solution (symmetric A)

$$\mathbf{x}(t) = \sum_{i=1}^{n} \mathbf{x}_i^T \, \mathbf{x}(0) \, \mathbf{x}_i \, e^{\lambda_i t} \tag{22}$$

Superposition and Time-Invariance

Linear time-invariant (LTI) *homogeneous* ODE systems satisfy the following useful properties:

LTI ODE

1. **Superposition:** If $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are solutions, then $\mathbf{A}\ \mathbf{x}(t) + \mathbf{B}\ \mathbf{y}(t)$ must also be solutions for any constant $\mathbf{A}$ and $\mathbf{B}$.
2. **Time Invariance:** If $\mathbf{x}(t)$ is a solution, then so is $\mathbf{x}(t + \Delta t)$ for any time displacement Δt.

An important consequence is that solutions to LTI inhomogeneous ODEs are readily obtained from solutions to the homogeneous problem through *convolution*. This observation is the basis for extensive use of the *Laplace and Fourier transforms* to study and solve LTI problems in engineering.

Impulse Response and Convolution

Let $h(t)$ the *impulse response* of a LTI system to a delta Dirac function at time zero:

$$\frac{dh}{dt} = \mathcal{L}(h) + \delta(t) \tag{23}$$

then, owing to the principle of superposition and time invariance, the response $u(t)$ to an arbitrary stimulus over time $f(t)$

$$\frac{du}{dt} = \mathcal{L}(u) + f(t) \tag{24}$$

is given by:

Convolution

$$u(t) = \int_{-\infty}^{+\infty} f(\theta)\ h(t-\theta)\ d\theta. \tag{25}$$

Fourier Transfer Function

Linear convolution in the time domain (25)

$$u(t) = \int_{-\infty}^{+\infty} f(\theta)\; h(t-\theta)\; d\theta$$

transforms to a linear product in the Fourier domain:

$$U(j\omega) = F(j\omega)\; H(j\omega) \tag{26}$$

where

$$U(j\omega) = \mathcal{F}(u(t)) = \int_{-\infty}^{+\infty} u(\theta)\; e^{-j\omega\theta}\; d\theta \tag{27}$$

is the Fourier transform of u.
The *transfer function* $H(j\omega)$ is the Fourier transform of the *impulse response* $h(t)$.

Laplace Transfer Function

For *causal systems*

$$h(t) \equiv 0 \quad \text{for} \quad t < 0 \tag{28}$$

the identical product form (26)

$$U(s) = F(s)\ H(s) \tag{29}$$

holds also for the Laplace transform

$$U(s) = \mathcal{L}(u(t)) = \int_0^{+\infty} u(\theta)\ e^{-s\theta}\ d\theta \tag{30}$$

where $s = j\omega$.

Bibliography

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