From Lumped to Continuum Model of a Non-myelinated Axon

Consider the voltage distribution \( v(x, t) \) along the length \( L \) of a non-myelinated axon over time. The axon has line resistance \( r \), and line capacitance \( c \). Assume initially the axon is at uniform voltage, \( v(x, 0) = v_0 \) for \( 0 < x < L \). Also, on each end of the axon the voltage is always zero, \( v(0, t) = v(L, t) = 0 \).

As in the Lecture 5 notes, we model this axon as a passive cable using a lumped circuit model with \( n \) segments, each of identical length \( \Delta x = L/n \), and with corresponding node voltages \( v_k(t) = v(k\Delta x, t) \), as shown in Figure 1. Here we investigate the effect of the number of cable segments \( n \) on the quality of the lumped approximation.

![Figure 1: Passive cable lumped model with \( n \) segments of length \( \Delta x = L/n \).](image)

a (10 pts)

Draw the lumped model for \( n = 4 \). Then using Kirchhoff’s laws (https://en.wikipedia.org/wiki/Kirchhoff%27s_circuit_laws) and Ohm’s Law (https://en.wikipedia.org/wiki/Ohm%27s_law) write out a set of three ODEs in the node voltages \( v_1(t) \), \( v_2(t) \), and \( v_3(t) \). What can you say about \( v_0(t) \) and \( v_4(t) \)?

b (15 pts)

Write the set of ODEs into matrix form \( \frac{du}{dt} = A u \) where \( u(t) \) is a column vector with components \( v_1(t) \), \( v_2(t) \), and \( v_3(t) \). Identify the coupling matrix \( A \), and find its eigenvalues \( \lambda_k \) and corresponding eigenvectors \( U_k \). Can the eigenvalues \( \lambda_k \) be complex, and why? And what can you say about the dot products between pairs of eigenvectors \( U_k \cdot U_l \)?

c (20 pts)

Find the general solution \( u(t) \) as a linear combination of eigenmodes \( u(t) = \sum_{l=1}^{3} c_l u_l(t) \) through separation of variables. Show that the particular solution can be found by expressing the constants \( c_l \) as dot products between the initial conditions \( u(0) = u_0 \) and the corresponding eigenvectors (principal axes) \( U_l \). Find the particular solution for initial conditions \( v_k(0) = v_0 \).

d (10 pts)

Now write out the set of ODEs for general \( n \) in the lumped circuit model \( v_k(t) = v(k\Delta x, t) \) where \( k = 1, 2, \ldots n-1 \). Further write out the Euler approximation of these ODEs with fixed time step size \( \Delta t \): \( t = i\Delta t \) for \( i = 0, 1, \ldots \infty \). This defines the finite difference approximate solution to the voltage in continuous space and time, \( v(x, t) \), as a recursion \( i \rightarrow i + 1 \) in a series of values on the discrete space-time grid \( v(k\Delta x, i\Delta t) \).

e (20 pts)

Plot the finite difference approximation of \( v(x, t) \) from initial conditions \( v(x, 0) = v_0 \) for \( n = 100 \), from \( t = 0 \) to \( 10 \) s. Choose an appropriate time step size \( \Delta t \) for the following parameters:
f (15 pts)

Convert the ODEs for the lumped circuit model $v_k(t) = v(k\Delta x, t)$ into a PDE for the continuum $v(x, t)$ by taking the limit as $\Delta x$ approaches zero (the number of segments $n$ approaches infinity). Write this PDE in the form of the diffusion equation, and identify the diffusivity $D$ in terms of the axon physical parameters. How do the eigenmodes of the lumped circuit model relate to the eigenmodes of the continuum model?

g (10 pts)

Finally, use pdepe to plot the solution of the PDE in $v(x, t)$ from the initial conditions $v(x, 0) = v_0$, and compare the results with your finite difference approximation in part e.